Markov fractions and slopes of exceptional bundles on $\mathbb{P}^2$

A.P. Veselov
Loughborough University, UK

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Plan

- Markov fractions and their properties
- Exceptional vector bundles on $\mathbb{P}^2$
- Markov fractions as exceptional slopes
- Discussion

Reference

A.P. Veselov *Markov fractions and the slopes of the exceptional bundles on $\mathbb{P}^2$.* arXiv:2501.06779.

Conway topograph and Farey fractions

The *Conway topograph* (**J.H. Conway 1991**) consists of the planar domains which are connected components of the complement to a trivalent tree imbedded in the plane. These domains were originally labelled by the "lax vectors" in the integer lattice $\mathbb{Z}^2$, but can be equivalently labelled by the rationals using *Farey mediant*

$$\frac{a_1}{b_1} \oplus \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}.$$

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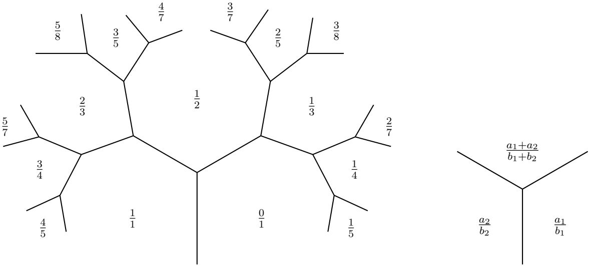

Figure: The Conway-Farey tree of fractions between 0 and 1.

The *Markov fraction tree* is the modification of the Conway-Farey tree, where the Farey mediant is replaced by the *Springborn mediant*

$$\frac{p_1}{q_1} * \frac{p_2}{q_2} = \frac{p_1 q_1 + p_2 q_2}{q_1^2 + q_2^2},$$

or, in the reduced form,

$$\frac{p_1}{q_1} * \frac{p_2}{q_2} = \frac{p}{q}, \quad p = \frac{p_1 q_1 + p_2 q_2}{p_2 q_1 - p_1 q_2}, \ \ q = \frac{q_1^2 + q_2^2}{p_2 q_1 - p_1 q_2}.$$

Markov fractions

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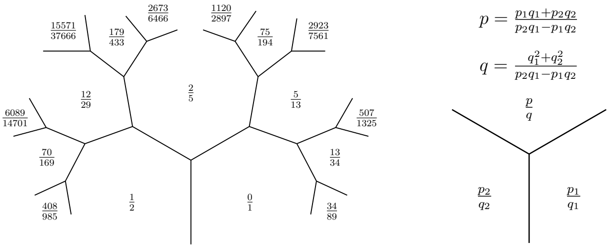

Figure: The Markov fraction tree with the Springborn local rule.

By definition, the Markov fractions between 0 and 1/2 (denoted as $\mathcal{MF}_R$) are defined recursively using the Springborn rule, starting from $\frac{0}{1}$ and $\frac{1}{2}$. Juxtaposition with the Conway-Farey tree establishes the Springborn bijection

$$\mu : \mathbb{Q} \cap [0, 1] \to \mathcal{MF}_R,$$

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By definition the function $\mu(x)$ satisfies the property

$$\mu\left(\frac{a}{b} \oplus \frac{c}{d}\right) = \mu\left(\frac{a}{b}\right) * \mu\left(\frac{c}{d}\right), \quad |ad - bc| = 1,$$

intertwining Farey and Sprinborn mediants of neighbours on the Farey tree. The set of all *Markov fractions* is defined as

$$\mathcal{MF} := \{n \pm \frac{p}{q}, \quad \frac{p}{q} \in \mathcal{MF}_R, \; n \in \mathbb{Z}\}.$$

The reduced set $\mathcal{MF}_R = \mathcal{MF} \cap [0, 1/2]$ is a fundamental domain of the natural action on $\mathcal{MF}$ of the integer affine group $Aff_1(\mathbb{Z})$: $x \to n \pm x$.

Markov fraction tree

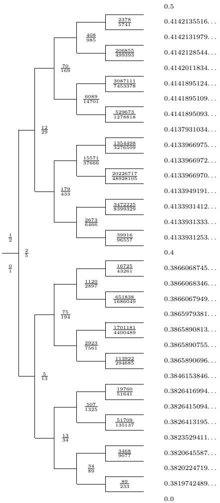

Fig. 3. Markov fractions in the interval $[0, \frac{1}{2}]$. Numerical values are shown in the right column.

Markov triples

Markov triples (**Markov 1880**) are the positive integer solutions of the *Markov equation*

$$q_1^2 + q_2^2 + q_3^2 = 3q_1q_2q_3.$$

They can be found from the obvious solution $(1, 1, 1)$ by applying permutations and Vieta involution $(q_1, q_2, q_3) \to (q_1, q_2, q_3')$ where

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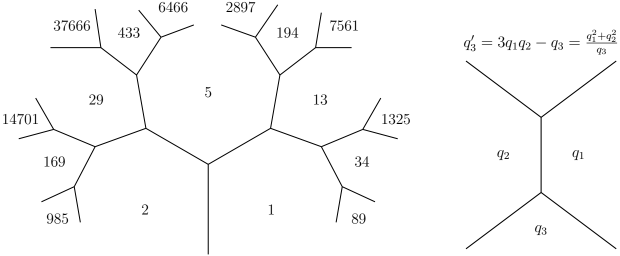

Figure: Markov numbers on the Conway topograph.

Markov spectrum of real numbers

Markov constant $\mu(\alpha)$ is a measure of irrationality of $\alpha \in \mathbb{R}$ defined as

$$\mu(\alpha) := \liminf_{b \to \infty} \left(b^2 \min_{a \in \mathbb{Z}} \left| \alpha - \frac{a}{b} \right| \right).$$

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Markov (1880): Lagrange spectrum above 1/3 is discrete and consists of

$$\mu = \frac{m}{\sqrt{9m^2-4}},$$

where m is one of the Markov numbers:

$$m = 1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985, \ldots$$

In other words, Markov triples describe the "most irrational numbers"

$$\alpha = \frac{b}{q_1} + \frac{q_2}{q_1 q_3} - \frac{3}{2} + \frac{\sqrt{9q_3^2-4}}{2q_3}, \quad bq_2 - aq_1 = q_3.$$

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Unicity Conjecture (Frobenius, 1913) *Every Markov number appears as maximal only in one Markov triple (so Markov spectrum is simple).*

Markov fractions and Markov numbers

Springborn 2024 introduced the Markov fractions as "the worst approximable rational numbers" with corresponding

$$C\left(\frac{p}{q}\right) := \inf_{\frac{a}{b}\in\mathbb{Q}\setminus\left\{\frac{p}{q}\right\}} b^2\left|\frac{p}{q}-\frac{a}{b}\right| \geq \frac{1}{3}.$$

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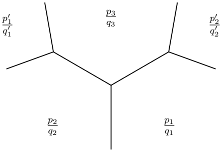
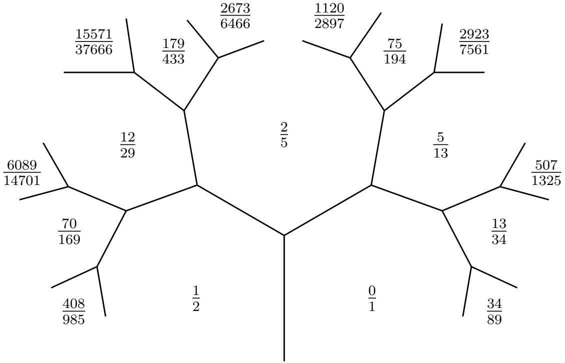


Key Lemma. The numbers on Markov fraction tree satisfy the relations

$$p_2q_3 - p_3q_2 = q_1, \quad p_3q_1 - p_1q_3 = q_2, \quad p_2q_1 - p_1q_2 = \frac{q_1^2+q_2^2}{q_3},$$

$$p_1' = \frac{p_2q_2+p_3q_3}{q_1},\ q_1' = \frac{q_2^2+q_3^2}{q_1}, \quad p_2' = \frac{p_1q_1+p_3q_3}{q_2},\ q_2' = \frac{q_1^2+q_3^2}{q_2}.$$

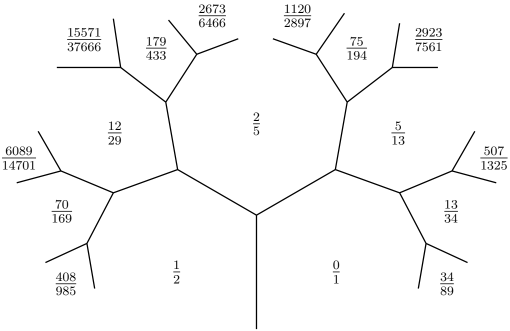
The denominators q of Markov fractions are Markov numbers.



The bottom right branch of Markov fraction tree: $\frac{1}{2}, \frac{2}{5}, \frac{5}{13}, \frac{13}{34}, \frac{34}{89}, \frac{89}{233} \dots$ consists of the ratios $p_k/q_k = F_{2k}/F_{2k+2}$ of the consecutive even Fibonacci numbers

$$F_k = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots.$$

Bottom left branch: Pell fractions



Pell numbers (x_n, y_n) are solutions of Pell's equations $x^2 - 2y^2 = (-1)^n$:

$(1,1), (3,2), (7,5), 17, 12), (41,29), (99,70), (239,169), (577,408), (1393,985), \dots$

The corresponding Markov-Pell fractions are the ratios $p_k/q_k = y_{2k}/y_{2k+1}$ of the consecutive Pell numbers defined by $y_{n+1} = 2y_n + y_{n-1}, \quad y_1 = 1, y_2 = 2,$

$$y_k = 1,\ 2, 5,\ 12, 29,\ 70, 169,\ 408, 985, \dots.$$

Springborn function

The *Springborn function* $\mu(x)$ is defined by the property

$$\mu\left(\frac{a}{b} \oplus \frac{c}{d}\right) = \mu\left(\frac{a}{b}\right) * \mu\left(\frac{c}{d}\right), \quad |ad - bc| = 1.$$

Springborn 2024: The extension of this function to real $x \in [0, 1]$ is continuous at irrational x and at rational x with $\mu(x) = p/q$ has jump $l = 3 - \sqrt{9 - 4/q^2}$.

In particular, $I = \left[\frac{p}{q} - \frac{1}{2}l(q), \frac{p}{q} + \frac{1}{2}l(q)\right]$ is the maximal interval around p/q, which is free of other Markov fractions.

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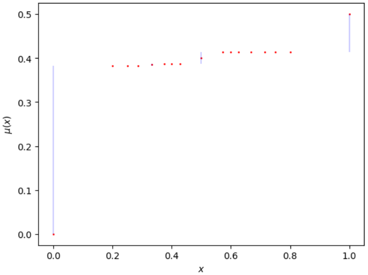

Figure: The first few values of $\mu(x)$ with vertical lines indicating the jumps

Saltus representation

APV 2025: Springborn function coincides with its *saltus function*:

$$\mu(x) = s_\mu(x) := -\frac{1}{2} I(1) + \sum_{a/b \in \mathbb{Q} \cap [0,1]} I\left(q\left(\frac{a}{b}\right)\right) H\left(x - \frac{a}{b}\right),$$

where $q(\frac{a}{b})$ is the Markov number and $H(x)$ is the Heaviside step function.

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The proof follows from the *McShane identity*, which can be written as

$$\frac{1}{2}(I(1) + I(2)) + \sum_{q \in \mathcal{M},\, q>2} I(q) = \frac{1}{2},$$

where $\mathcal{M}$ is the set of all Markov numbers.

Corollary

The derivative $\mu'(x) = 0$ *almost everywhere.*

Let E be an algebraic vector bundle on complex projective plane $\mathbb{P}^2$ of rank r with the Chern classes c_1 and c_2. Since $H^2(\mathbb{P}^2) \cong \mathbb{Z}$ and $H^4(\mathbb{P}^2) \cong \mathbb{Z}$ we can consider c_1 and c_2 as integers. The **slope** of E is defined as the ratio

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Exceptional vector bundles on $\mathbb{P}^2$

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The bundle E is called **stable** if for any proper sub-sheaf $F \subset E$ we have

$$\mu(F) < \mu(E)$$

and **rigid** if $Ext^1(E, E) = 0$. The vector bundles, which are both stable and rigid, are called **exceptional.** They can also be defined by the conditions

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Drèzet and Le Potier 1984: The exceptional vector bundles E on $\mathbb{P}^2$ are uniquely determines by the slope $\mu(E)$.

The main question is to describe the set $\mathfrak{E}$ of all possible slopes of the exceptional bundles (called **exceptional slopes**).

Exceptional slopes and Markov fractions

Drèzet and Le Potier 1984: The set $\mathfrak{E}$ is the image of the special function $\epsilon : \mathfrak{D} \to \mathbb{Q}$, where $\mathfrak{D}$ is the set of dyadic (binary) rationals $m/2^n$.

This function has the properties $\epsilon(-x) = -\epsilon(x), \quad \epsilon(x+n) = \epsilon(x) + n,\ n \in \mathbb{Z}$ and is uniquely defined by the condition: if $\epsilon\left(\frac{m}{2^n}\right) = \frac{p_1}{q_1},\ \ \epsilon\left(\frac{m+1}{2^n}\right) = \frac{p_2}{q_2}$, then

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APV 2025: The set $\mathfrak{E}$ of slopes of the exceptional bundles on $\mathbb{P}^2$ coincides with the set of all Markov fractions.

The key observation is that the Drèzet-Le Potier defining relation for $p_1/q_1 < p_2/q_2$ is equivalent to the Springborn mediant rule:

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Corollary (Rudakov 1988) The ranks of the exceptional vector bundles on $\mathbb{P}^2$ are Markov numbers.

Unicity conjecture

The celebrated *Unicity conjecture* claims that any Markov triple is uniquely determined by its maximal part. Springborn reformulated it as the following

Conjecture 1. For any Markov number q there exists unique Markov fraction $0 \leq \frac{p}{q} \leq \frac{1}{2}$.

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Note that for any Markov fraction p/q we have the quadratic congruence

$$p^2 + 1 \equiv 0 \, (mod \;\; q).$$

For prime q it has a unique (up to a sign) solution, so both conjectures hold true in this case (**Baragar 1996**).

For example, for Markov fraction $\frac{15571}{37666}$ with $37666 = 2 \times 37 \times 509$ the congruence $x^2 + 1 \equiv 0 \, (mod \;\; 37666)$ has 4 solutions $x \equiv \pm 2337, \pm 15571$.

Can we "characterise" the particular solution given by the numerator p of the Markov fraction for composite q?

Second Chern class and Markov form

By Riemann-Roch theorem the second Chern class $c_2(E)$ of the exceptional bundle E can be computed in terms of $p = c_1(E)$, $q = r(E)$ as

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It is interesting that the *Markov binary quadratic form* can be written in these terms as

$$f(x,y) = qx^2 + (3q-2p)xy + (s-3p)y^2.$$

The roots of the quadratic equation

$$f(\alpha,1) = q\alpha^2 + (3q-2p)\alpha + (s-3p) = 0$$

are Markov irrationalities, which are limit points of the set of Markov fractions.

Rudakov 1988: the exceptional collections on quadrics $\mathbb{P}^1 \times \mathbb{P}^1$ have ranks (x, y, z) satisfying the Diophantine equation

$$x^2 + y^2 + 2z^2 = 4xyz$$

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Karpov and Nogin 1989: generalisation to other del Pezzo surfaces S known to be isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, or to X_m being the plane $\mathbb{P}^2$ blown up in m generic points with $0 \leq m \leq 8$. In particular, for X_3 we have the Diophantine equation

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What are the corresponding exceptional slopes in the del Pezzo cases?

A.A. Markoff *Sur les formes quadratiques binaires indéfinies.* Mathematische Annalen, **15** (1879), 381-406; **17** (1880), 379-399.

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