Elliptic Automorphic Lie Algebras and Integrable Systems

Casper Oelen

Maxwell Institute/Heriot-Watt University

Algebraic geometry, integrable systems and automorphic forms

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Based on joint work with Sara Lombardo and Vincent Knibbeler

Overview of talk

- What are automorphic Lie algebras?
- Motivation from Integrable Systems
- Results related to elliptic automorphic Lie algebras[1, 2]
  - Holod's hidden symmetry algebra of the Landau-Lifshitz equation
  - Uglov's algebra

[1] Vincent Knibbeler, Sara Lombardo, and Casper Oelen. "A classification of automorphic Lie algebras on complex tori". In: *Proceedings of the Edinburgh Mathematical Society* (2024), pp. 1–43.

[2] Sara Lombardo and Casper Oelen. "Normal forms of elliptic automorphic Lie algebras and Landau-Lifshitz type of equations". In: *arXiv preprint arXiv:2412.20482* (2024).

What are automorphic Lie algebras?

Automorphic Lie algebras (aLias) are Lie algebras of meromorphic maps

$$\text{Riemann surface } X \to \text{Lie algebra } \mathfrak{g}$$

($\mathfrak{g}$ finite-dimensional complex) with the following properties:

1. pointwise Lie bracket: $[f,g](p) = [f(p), g(p)]$, $p \in X$
2. holomorphic outside a set of "punctures"
3. equivariant with respect to a group Γ acting on X and $\mathfrak{g}$ by automorphisms

Motivation

- ALias generalise various Lie algebras
  - (twisted) current algebras
  - (twisted) loop algebras
  - Onsager algebras
- Appear in integrable systems
  - In the construction and classification of classical integrable systems
- Related to *geometric deep learning*[3]
- In algebra: example of *equivariant map algebras*[4]

[3] Vincent Knibbeler. "Computing equivariant matrices on homogeneous spaces for geometric deep learning and automorphic Lie algebras". In: *Advances in Computational Mathematics* 50.2 (2024), p. 27.

[4] Erhard Neher, Alistair Savage, and Prasad Senesi. "Irreducible finite-dimensional representations of equivariant map algebras". In: *Transactions of the American Mathematical Society* 364.5 (2012), pp. 2619–2646.

Twisted loop algebras

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra (simple) and $\mathbb{C}[z, z^{-1}]$ the ring of Laurent polynomials. Let ρ be an order n automorphism of $\mathfrak{g}$.

- Form the *loop algebra*

$$\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$$

  with bracket $[A \otimes f, B \otimes g] := [A, B] \otimes fg$.
- $\mathcal{L}(\mathfrak{g})$ is the Lie algebra of regular maps $f : \mathbb{C} \setminus \{0\} \to \mathfrak{g}$.
- The *twisted* loop algebra $\mathcal{L}(\mathfrak{g}, \rho)$ is the subalgebra of equivariant maps $f : \mathbb{C} \setminus \{0\} \to \mathfrak{g}$ such that

$$\rho f(z) = f(\epsilon z), \quad \epsilon^n = 1.$$

Kac (1969): For any inner automorphism ρ, there is an isomorphism

$$\mathcal{L}(\mathfrak{g}, \rho) \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$$

of $\mathbb{Z}$-graded Lie algebras.

The ingredients of an aLia are:

- Finite-dimensional complex Lie algebra $\mathfrak{g}$.
- Compact Riemann surface X. $(\leftrightarrow \mathbb{C}_\infty)$
- Finite group Γ acting on $\mathfrak{g}$ and on X via the homomorphisms

$$\rho : \Gamma \to \mathrm{Aut}(\mathfrak{g}), \quad \sigma : \Gamma \to \mathrm{Aut}(X). \quad (\leftrightarrow \Gamma = C_n,\ \sigma(r)z = \epsilon z)$$

- Ring $\mathcal{O}_{\mathbb{X}}$ of regular functions on $\mathbb{X} := X \setminus \sigma(\Gamma)S$, $S \subset X$.
  $(\leftrightarrow S = \{0, \infty\},\ \mathcal{O}_{\mathbb{X}} = \mathbb{C}[z, z^{-1}])$

Definition aLias

An aLia $\mathfrak{A}$ is a fixed point Lie subalgebra of $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}}$ with respect to the action

$$\gamma \cdot (A \otimes f(z)) = \rho(\gamma)A \otimes f(\sigma(\gamma^{-1})z), \quad \gamma \in \Gamma.$$

That is,

$$\mathfrak{A} = (\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}})^{\rho \otimes \sigma(\Gamma)} = \{a \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}} : \gamma \cdot a = a \quad \forall \gamma \in \Gamma\}.$$

Equivalently,

$$a(z) \in \mathfrak{A} \iff a(\sigma(\gamma)z) = \rho(\gamma)a(z) \quad \forall \gamma \in \Gamma.$$

History of aLias

- ALias as a subject on its own was introduced by Lombardo and Mikhailov in [LM04],[LM05]. Further work related to integrable systems by Bury and Mikhailov [BM21].
- Algebraic development by Lombardo and Sanders [LS10], and later with Knibbeler [KLS17], [KLS20], with Veselov [KLV23], with CO [KLO24], and more recently by Knibbeler [Kni25].
- Representation theory studied by Knibbeler and Lombardo with Duffield [DKL24].
- In algebra, aLias are known as equivariant map algebras, introduced by Neher, Savage and Senesi [NSS12].

The classification of aLias is part of the program of classifying Lax operators and hence of classifying (classical) integrable systems.

Example 1: Elliptic aLia with $\Gamma = C_2$

Let us compute

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2}$$

with $C_2 = \langle \gamma \rangle$ and on a complex torus $T = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ such that

- $\gamma \cdot z = -z$.
- $\gamma \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}$.
- $\mathbb{T} = T \setminus C_2 \cdot \{0\} = T \setminus \{0\}$.

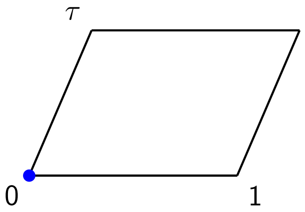

Figure: Complex torus $T = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and $S = \{0\}$.

Example 1: Elliptic aLia with $\Gamma = C_2$

Let $\mathcal{M}(T)$ be the field of meromorphic functions on T. To describe the ring

$$\mathcal{O}_{\mathbb{T}} := \{f \in \mathcal{M}(T) : f \text{ has poles restricted to } C_2 \cdot \{0\} = \{0\}\},$$

we use the *Weierstrass $\wp$-function* associated to lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$:

$$\wp_\Lambda : \mathbb{C}/\Lambda \to \mathbb{C} \cup \{\infty\}, \quad \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

- Meromorphic with a (double) pole at $z = 0$.
- $\mathcal{M}(T) = \mathbb{C}(\wp_\Lambda, \wp'_\Lambda)$.
- $\wp_\Lambda(-z) = \wp_\Lambda(z)$.
- $(\wp'_\Lambda)^2 = 4\wp_\Lambda^3 - g_2(\tau)\wp_\Lambda - g_3(\tau)$.

$$\mathcal{O}_{\mathbb{T}} = \mathbb{C}[\wp_\Lambda, \wp'_\Lambda] = \mathbb{C}[\wp_\Lambda] \oplus \wp'_\Lambda \mathbb{C}[\wp_\Lambda]$$

Example 1: Elliptic aLia with $\Gamma = C_2$

Eigenspace decomposition:

$$\mathfrak{sl}_2 = \mathfrak{sl}_2^+ \oplus \mathfrak{sl}_2^- = \mathbb{C}\langle h, e, f \rangle = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle.$$

Futhermore, $\mathcal{O}_{\mathbb{T}} = \mathcal{O}_{\mathbb{T}}^+ \oplus \mathcal{O}_{\mathbb{T}}^- = \mathbb{C}[\wp] \oplus \wp'\mathbb{C}[\wp]$. We have

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2} = \mathfrak{sl}_2^+ \otimes \mathcal{O}_{\mathbb{T}}^+ \oplus \mathfrak{sl}_2^- \otimes \mathcal{O}_{\mathbb{T}}^- = \mathbb{C}\langle H, E, F \rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp],$$

where $H = h \otimes 1$, $E = e \otimes \wp'$, $F = f \otimes \wp'$. Lie structure:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H \otimes (4\wp^3 - g_2(\tau)\wp - g_3(\tau)).$$

Note for example that

$$E(\gamma \cdot z) = e \otimes \wp'(-z) = -e \otimes \wp'(z) = \gamma \cdot E(z).$$

Observe:

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2} \ncong \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{C_2}$$

ALias and integrable systems

Consider a nonlinear PDE

$$u_t = N(u), \quad u = u(x,t). \tag{1}$$

If we can find a Lax pair for (1), that is, operators L, M such that

$$u_t = N(u) \iff L_t = [L, M],$$

where $[L, M] = LM - ML$, then solvable via Inverse Scattering Transform.

- Rare property for nonlinear PDEs. Our definition of integrable.
- *Wahlquist-Estabrook prolongation method* $\rightsquigarrow$ Lax pair.

Reversed direction:

general form of Lax pair $\rightsquigarrow$ integrable equations

Via the method of *reduction*.

Consistency between a pair of linear equations for a vector function $\psi = \psi(x, t; \lambda)$:

$$\begin{cases} L\psi = \psi_x - X\psi = 0, \\ M\psi = \psi_t - T\psi = 0, \end{cases}$$

where X, T are matrices depending on x, t and a complex parameter λ.

Integrable nonlinear equation $\iff X_t - T_x + [X, T] = 0.$

Example: the KdV equation

$$u_t = 6uu_x + u_{xxx}, \quad u = u(t, x),$$

can be written as the consistency condition of the linear system

$$\psi_x = X\psi, \quad \psi_t = T\psi,$$

where ψ is a vector and

$$X = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -u_x & 4\lambda + 2u \\ 4\lambda^2 - 2\lambda u - 2u^2 - u_{xx} & u_x \end{pmatrix},$$

where λ is the *spectral parameter*.

Application I: reduction of Lax pairs

Suppose λ is a rational parameter. Consider the "fairly general" $\mathfrak{sl}(N, \mathbb{C})$-valued Lax pair

$$L(x,t;\lambda) = \partial_x - X(x,t;\lambda), \quad X = Q_0 + Q\lambda + \bar{Q}\lambda^{-1},$$
$$M(x,t;\lambda) = \partial_t - T(x,t;\lambda), \quad T = P_0 + P\lambda + \bar{P}\lambda^{-1} + Q^2\lambda^2 + \bar{Q}^2\lambda^{-2},$$

where $Q_0, Q, \bar{Q}, P_0, P, \bar{P} \in \mathfrak{sl}(N, \mathbb{C})$.

- The compatibility condition

$$X_t - T_x + [X, T] = 0$$

  yields a system of $(5(N^2-1))$ nonlinear coupled equations for the matrix entries.
- By definition integrable.
- $X, T \in \mathfrak{sl}(N, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda, \lambda^{-1}]$.
- By imposing *reductions* on $L(\lambda)$, $M(\lambda)$ one obtains integrable PDEs.

Example 2: D_N reduction

Following [Lom04], consider

$$r : L(\lambda) \mapsto SL(\omega\lambda)S^{-1}, \quad s : L(\lambda) \mapsto -L^A(1/\lambda),$$

where $S_{ij} = \delta_{ij}\omega^i$, $\omega = e^{2\pi i/N}$, and where L^A is the formal adjoint of X. Impose invariance of $\Gamma = \langle r, s\rangle \cong D_N$ on $L(\lambda)$ and $M(\lambda)$.

- $L(\lambda), M(\lambda) \in (\mathfrak{sl}(N,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda, \lambda^{-1}])^{D_N}$.
- Compatibility condition reduces to

$$Q_t - P_x + [\bar{Q}, Q^2] = 0, \quad Q^2_x = [Q, P].$$

- In components

$$q_{it} - p_{ix} + q_i q^2_{i+1} - q^2_{i-1} q_i = 0, \quad q_i p_{i+1} - p_i q_{i+1} - (q_i q_{i+1})_x = 0.$$

This results in

$$u_{1t} = \frac{1}{3}(u_{3xx} - u_{2xx}) + \frac{1}{3}u_{1x}(u_{3x} - u_{2x}) - \exp(2u_2) + \exp(2u_3)$$

and cyclic permutations of the indices $1, 2, 3$.

Application II: Lax pairs on higher genus curves

Suppose that the zero-curvature equation

$$\partial_t X - \partial_x T + [X, T] = 0,$$

where $X(x,t;\lambda)$ and $T(x,t;\lambda)$ are *rational* matrix functions of a spectral parameter λ,

$$X = u_0(x,t) + \sum_{i,s} u_{is}(x,t)(\lambda-\lambda_i)^{-s}, \quad T = v_0(x,t) + \sum_{j,k} v_{jk}(x,t)(\lambda-\mu_j)^{-k}.$$

- No obstruction in obtaining a well-defined system of equations.
- Obstruction for "naive" generalisation to matrix functions meromorphic on algebraic curves of genus > 0.

Resolved by imposing a symmetry condition on L and M:

$$X(\gamma\cdot\lambda) = \gamma\cdot X(\lambda), \quad T(\gamma\cdot\lambda) = \gamma\cdot T(\lambda), \text{ where } \gamma \in \Gamma.$$

Examples with spectral parameter on genus 1 curves

1. Landau-Lifshitz equation: $S_t = S \times S_{xx} + S \times JS \iff [L(z), M(z)] = 0$,

$$L(z) = \frac{\partial}{\partial x} - i \sum_{\alpha=1}^{3} w_\alpha(z) S_\alpha \sigma_\alpha,$$

$$M(z) = \frac{\partial}{\partial t} - i \sum_{\alpha,\beta,\gamma=1}^{3} w_\alpha(z) \sigma_\alpha S_\beta S_{\gamma x} \epsilon_{\alpha\beta\gamma} + 2i w_1(z) w_2(z) w_3(z) \sum_{\alpha=1}^{3} w_\alpha(z)^{-1} S_\alpha \sigma_\alpha,$$

where the $w_i(z)$ are elliptic functions, $S(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))$ and σ_α are the Pauli matrices.

2. Krichever-Novikov equation: $v_t = \frac{1}{4} v_{xxx} - \frac{3}{8} \frac{v_{xx}^2}{v_x} + \frac{3}{8} \frac{4v^3 - g_2 v - g_3}{v_x}$. Lax pair:

$$L(z) = \frac{\partial}{\partial x} - \sum_{j=1}^{3} w_j(z) M_j \sigma_j, \quad M(z) = \frac{\partial}{\partial t} - \sum_{j=1}^{3} \left[-\frac{d}{dz} w_j(z) M_j + w_j(z)(QM_j + P_j) \right] \sigma_j,$$

$$(z \in \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \quad M_1 = \frac{v^2 - 1}{2v_x}, \qquad M_2 = i \frac{v^2 + 1}{2v_x}, \qquad M_3 = -\frac{v}{v_x}.$$

$$Q = \frac{1}{2} \left(-\frac{1}{2} \frac{v_{xxx}}{v_x} - \frac{1}{4} \frac{v_{xx}^2}{v_x^2} + \frac{(A_1 - A_2)(v^4 + 1) - 6(A_1 + A_2)v^2}{v_x^2} \right),$$

$$P_1 = \frac{v_{xx} v}{v_x} - v_x, \qquad P_2 = -i \left(\frac{v_{xx}}{v_x} + v_x \right), \qquad P_3 = -\frac{v_{xx}}{2v_x^2}.$$

The classification of $(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma}$

Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The following Lie algebras appear in the classification with $\mathbb{T} = T \setminus \Gamma \cdot \{0\}$:

1. $$\mathfrak{E}_\tau = \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[x,y]/(y^2 - 4x^3 + g_2(\tau)x + g_3(\tau)),$$
   with Lie structure inherited from $\mathfrak{sl}_2(\mathbb{C})$.
2. $$\mathfrak{S}_\tau = \mathbb{C}\langle E, F, H\rangle \otimes_{\mathbb{C}} \mathbb{C}[x],$$
   with Lie structure (linear over $\mathbb{C}[x]$)
   $$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H \otimes (4x^3 - g_2(\tau)x - g_3(\tau)).$$
3. The Onsager algebra $\mathfrak{O}$ with basis A_k, G_m $(k \in \mathbb{Z}, m \in \mathbb{N})$ and brackets
   $$[A_k, A_l] = 4G_{k-l}, \quad [A_k, G_m] = 2(A_{k-m} - A_{k+m}), \quad [G_m, G_n] = 0,$$
   with $G_{-m} = -G_m$ $(m > 0)$ and $G_0 = 0$.

The Lie algebras that appear in the classification of

$$(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma} \qquad (\mathbb{T} = T \setminus \Gamma \cdot \{0\})$$

fall into three (pairwise non-isomorphic) classes determined by the branch points of the canonical projection

$$\pi : \mathbb{T} \to \mathbb{T}/\Gamma.$$

# branch points	Lie algebra
0	$\mathfrak{C}_{[\tau]}$
2	$\mathfrak{O}$
3	$\mathfrak{S}_{[\tau]}$

Table: Lie algebra associated to the number of branch points of the quotient map $\mathbb{T} \to \mathbb{T}/\Gamma$.

- $\mathfrak{C}_{[\tau]} \cong \mathfrak{C}_{[\tau']} \iff [\tau] = [\tau']$ in $SL(2,\mathbb{Z})\backslash\mathbb{H}$
- $[\tau] = [\tau'] \implies \mathfrak{S}_{[\tau]} \cong \mathfrak{S}_{[\tau']}$

Two aLias with distinct $\sigma : D_2 \to \mathrm{Aut}(T)$

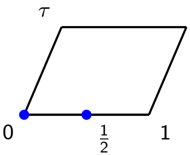

Figure: $\sigma(D_2) = \langle r, s\rangle$ with $r(z) = z + \frac{1}{2}$ and $s(z) = -z$. $S = \{0\}$.

- $(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\sigma(D_2)} \ncong \mathfrak{sl}(2, R)$, for any ring R.

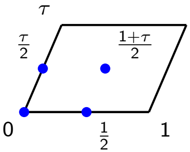

Figure: $\sigma(D_2) = \langle r, s\rangle$ with $r(z) = z + \frac{1}{2}$ and $s(z) = z + \frac{\tau}{2}$. $S = \{0\}$.

- $(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\sigma(D_2)} \cong \mathfrak{sl}(2, R)$, with $R = \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}]$.

Example 3: Landau-Lifshitz equation

The aLia with $\Gamma = D_2$ plays a prominent role in integrable systems:

- Appears in Sklyanin's Lax pair for the Landau-Lifshitz equation.

The (fully anisotropic) Landau-Lifshitz (LL) equation

$$S_t = S \times S_{xx} + S \times JS,$$

$S = (S_1(x,t), S_2(x,t), S_3(x,t))$, $J = \mathrm{diag}(J_1, J_2, J_3)$ ($J_\alpha \neq J_\beta$ for $\alpha \neq \beta$) can be written as the compatibility condition $[L, M] = 0$ where

$$L(z) = \frac{\partial}{\partial x} - i \sum_{\alpha=1}^{3} w_\alpha(z) S_\alpha \sigma_\alpha,$$

$$M(z) = \frac{\partial}{\partial t} - i \sum_{\alpha,\beta,\gamma=1}^{3} w_\alpha(z) \sigma_\alpha S_\beta S_{\gamma x} \epsilon^{\alpha\beta\gamma} + 2i w_1(z) w_2(z) w_3(z) \sum_{\alpha=1}^{3} w_\alpha(z)^{-1} S_\alpha \sigma_\alpha,$$

where the $w_\alpha(z)$ satisfy $w_\alpha(z)^2 - w_\beta(z)^2 = \frac{1}{4}(J_\alpha - J_\beta)$, and σ_α are the Pauli matrices.

Example 3: Landau-Lifshitz equation

Let $T = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and consider $\sigma : D_2 \to \mathrm{Aut}(T)$ be defined by

$$\sigma(r_1)z = z + \tfrac{1}{2}, \quad \sigma(r_2)z = z + \tfrac{\tau}{2},$$

and $\rho : D_2 \to \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{C}))$

$$\rho(r_1) = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(r_2) = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For Lax pair of LL equation: $L, M \in (\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(D_2)}$, i.e.

$$L(\sigma(r)z) = \rho(r)L(z), \quad M(\sigma(r)z) = \rho(r)M(z).$$

Claim: Elements $A_\alpha := w_\alpha(z)\sigma_\alpha$ and $B_\alpha := w_\beta(z)w_\gamma(z)\sigma_\alpha$ $(\alpha = 1,2,3)$ form a basis of $(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2}$ over $\mathbb{C}[\wp_{\frac{1}{2}\Lambda}]$:

$$(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2} = \bigoplus_{\alpha=1}^{3} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}]A_\alpha \oplus \mathbb{C}[\wp_{\frac{1}{2}\Lambda}]B_\alpha.$$

The Wahlquist-Estabrook prolongation algebra of the LL equation is isomorphic to $\mathfrak{A}(D_2) \oplus \mathbb{C}^2$.

Suppose that $w_1(z), w_2(z), w_3(z)$ uniformise the complex curve

$$E_{r_1,r_2,r_3} : \begin{cases} \lambda_1^2 - \lambda_3^2 = r_3 - r_1, \\ \lambda_2^2 - \lambda_1^2 = r_1 - r_2, \end{cases}$$

with $r_i \neq r_j$ for $i \neq j$. Let

$$X_i = v_i \otimes w_i, \quad X_i' = v_i \otimes w_j w_k, \quad i = 1, 2, 3, \quad (\textit{different notation}!)$$

with $\mathbb{C}\langle v_1, v_2, v_3 \rangle \cong \mathfrak{sl}(2, \mathbb{C})$ and $[v_i, v_j] = \varepsilon_{ijk} v_k$. Let α_{ij} be the characters of D_2.

$$\begin{aligned} \left(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{T \setminus D_2 \cdot \{0\}}\right)^{D_2} &= \bigoplus_{i,j=0}^{1} \mathfrak{sl}(2,\mathbb{C})^{\alpha_{ij}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\overline{\alpha_{ij}}} \\ &= \bigoplus_{i \neq j \neq k \neq i} \mathbb{C} v_i \otimes \left(\mathbb{C}[\wp_{\frac{1}{2}\Lambda}] w_i \oplus \mathbb{C}[\wp_{\frac{1}{2}\Lambda}] w_j w_k\right) \\ &= \bigoplus_{i=1}^{3} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}] X_i \oplus \mathbb{C}[\wp_{\frac{1}{2}\Lambda}] X_i'. \end{aligned}$$

Corollary

$$\left(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{T \setminus D_2 \cdot \{0\}}\right)^{D_2} = \mathbb{C}\langle X_1, X_2, X_3 \rangle.$$

Theorem (Kac)

Let σ be an inner automorphism of order n of a simple finite-dimensional Lie algebra $\mathfrak{g}$ of the form

$$\sigma = \exp\left(\frac{2\pi i}{n}\mathrm{ad}(h)\right).$$

Then the automorphism $\Psi(z)$ of the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ defined by

$$\Psi(z) = \exp(\ln(z)\mathrm{ad}(h))$$

establishes an isomorphism between $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z^n, z^{-n}]$ and the twisted Lie algebra $\mathcal{L}(\mathfrak{g}, \sigma) = (\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}])^{C_n}$.

The proof follows from the relation

$$\Psi(e^{\frac{2\pi i}{n}} z) = \sigma\Psi(z).$$

Strategy: Find elliptic analogues to Ψ.

Any simple complex Lie algebra $\mathfrak{g}$ of rank ℓ with root system Φ has a basis, known as the Chevalley basis, given by $\{h_i, a_\alpha : i = 1, \ldots, \ell, \text{and } \alpha \in \Phi\}$ such that the brackets are given by

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, a_\alpha] &= \alpha(h_i) a_\alpha, \\ [a_\alpha, a_{-\alpha}] &= h_\alpha, \\ [a_\alpha, a_\beta] &= \pm(r+1) a_{\alpha+\beta}, \quad && \alpha + \beta \in \Phi, \\ [a_\alpha, a_\beta] &= 0, && \alpha + \beta \notin \Phi \cup \{0\}, \end{aligned}$$

where $\alpha, \beta \in \Phi$ and where r is a certain integer.

- Aim: we would like an analogues basis for the aLias $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}})^\Gamma$.

Normal forms of aLias

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, $T = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, and $S \subset T$ (nonempty, finite). Let

$$\mathfrak{A}(\mathfrak{g}, \tau, S) = (\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{T \setminus D_2 \cdot S})^{\rho \otimes \tilde{\sigma}(D_2)}.$$

Definition

The normal form for $\mathfrak{A}(\mathfrak{g}, \tau, S)$ is a basis analogues to the Chevalley basis of $\mathfrak{g}$:

$$\mathfrak{A}(\mathfrak{g}, \tau, S) = \mathbb{C}\langle \{H_i, A_\alpha : i = 1, \ldots, \ell, \text{and } \alpha \in \Phi\}\rangle \otimes_{\mathbb{C}} \mathcal{O}^{D_2}_{T \setminus D_2 \cdot S},$$

with Lie structure obtained from $\mathfrak{g}$ by replacing h_i with H_i and A_α with a_α, and keep the structure constants the same. The basis elements satisfy:

$$H_i(\sigma(\gamma)z) = \rho(\gamma)H_i(z), \quad A_\alpha(\sigma(\gamma)z) = \rho(\gamma)A_\alpha(z), \quad \forall \gamma \in D_2$$

for $i = 1, \ldots, \ell$ and $\alpha \in \Phi$.

Let $\xi_p(z) = w_1(z)w_1(z-p)$: D_2-invariant function on T with simple poles at $0, p$.

Theorem (Lombardo, Oelen ('25))

Let $\mathfrak{g}$ be a complex reductive Lie algebra. Let $\rho : D_2 \to \mathrm{Inn}(\mathfrak{g})$ be a representation that factors through a representation $\overline{\rho} : PGL(2,\mathbb{C}) \to \mathrm{Inn}(\mathfrak{g})$ and suppose that $\sigma : D_2 \to \mathrm{Aut}(T)$ is a faithful homomorphism that embeds D_2 as translations of T. Let $S = \{p_0 = 0, p_1, \ldots, p_{n-1}\}$, and $\mathbb{T} = T \setminus D_2 \cdot S$. The following isomorphism of Lie algebras holds:

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(D_2)} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}, \xi_{p_1}, \ldots, \xi_{p_{n-1}}].$$

When $\mathfrak{g}$ is simple, the automorphic Lie algebra $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(D_2)}$ has normal form

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(D_2)} = \mathbb{C}\langle\{\Omega_{\overline{\rho}} h_i, \Omega_{\overline{\rho}} a_\alpha : i = 1, \ldots, n, \text{ and } \alpha \in \Phi\}\rangle \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\tilde{\sigma}(D_2)},$$

where $\{h_i, a_\alpha : i = 1, \ldots, n, \text{ and } \alpha \in \Phi\}$ is a Chevalley basis for $\mathfrak{g}$.

Ingredients of proof: D_2-intertwiner

Let $\theta_j(z|\tau)$, $j = 1, \ldots, 4$ be the Jacobi theta functions and define

$$\Omega(z) = \begin{pmatrix} \theta_3(2z|2\tau) & \psi_-(z)\theta_2(2z|2\tau) \\ \theta_2(2z|2\tau) & \psi_+(z)\theta_3(2z|2\tau) \end{pmatrix},$$

where

$$\psi_\pm(z) = \pm\frac{\theta_4^2(0|\tau)}{\theta_3(0|\tau)}\frac{\theta_3(2z|\tau)}{\theta_1(2z|\tau)} - \frac{\theta_3^2(0|\tau)}{\theta_4(0|\tau)}\frac{\theta_4(2z|\tau)}{\theta_1(2z|\tau)}.$$

Then

1. $\Omega(z+\frac{1}{2}) = T_1\Omega(z)$, with $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2. $\Omega(z+\frac{\tau}{2}) = e^{-\pi i(2z+\frac{\tau}{2})}T_2\Omega(z)$, with $T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3. $\det\Omega(z) = -\theta_2^2(0|\tau)\theta_1(2z|\tau)$

Proposition

Let $\omega := \mathrm{Ad}(\Omega) \in \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})$. *Then*

$$\omega(z+\tfrac{1}{2}) = \mathrm{Ad}(T_1)\omega(z), \quad \omega(z+\tfrac{\tau}{2}) = \mathrm{Ad}(T_2)\omega(z).$$

Idea of proof

- Consider the holomorphic map $\Omega : \mathbb{C} \setminus \frac{1}{2}\Lambda \to GL(2,\mathbb{C})$

$$\Omega(z) = \begin{pmatrix} \theta_3(2z|2\tau) & \psi_-(z)\theta_2(2z|2\tau) \\ \theta_2(2z|2\tau) & \psi_+(z)\theta_3(2z|2\tau) \end{pmatrix}$$

- $\Omega'(z) := [\Omega(z)] \in PGL(2,\mathbb{C})$ descends to a D_2-equivariant map on T:

$$\Omega'(z + \tfrac{1}{2}) = [T_1]\Omega'(z), \quad \Omega'(z + \tfrac{\tau}{2}) = [T_2]\Omega'(z)$$

- Use $\overline{\rho} : PGL(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$ and let $\Omega_{\overline{\rho}}(z) = \overline{\rho}([\Omega(z)])$
- Fin. dim. irreps of $PGL(2,\mathbb{C})$: $\mathrm{Sym}^{2n}(\mathbb{C}^2) \otimes \det^{-n} \rightsquigarrow \Omega_{\overline{\rho}}(z)$ preserves location of poles
- $\Omega_{\overline{\rho}}$ is a D_2-equivariant automorphism of $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}$
- $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}} = \Omega_{\overline{\rho}}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}$ and hence

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2} = \Omega_{\overline{\rho}}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{D_2} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{D_2}$$

- Finally, $\mathcal{O}_{\mathbb{T}}^{D_2} = \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}, \xi_{p_1}, \ldots, \xi_{p_{n-1}}]$

The explicit generators of $(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2}$

Write θ_j for $\theta_j(0|\tau)$, $j = 1, 2, 3, 4$.

$$H(z) = \begin{pmatrix} \theta_2^2\mu_2(z)\mu_3(z) & -\theta_4^2\mu_1(z)\mu_2(z) - \theta_3^2\mu_1(z)\mu_3(z) \\ -\theta_4^2\mu_1(z)\mu_2(z) + \theta_3^2\mu_1(z)\mu_3(z) & -\theta_2^2\mu_2(z)\mu_3(z) \end{pmatrix},$$

$$E(z) = \frac{1}{2}\begin{pmatrix} \mu_1(z) & -\frac{\theta_3^2}{\theta_2^2}\mu_2(z) - \frac{\theta_4^2}{\theta_2^2}\mu_3(z) \\ \frac{\theta_3^2}{\theta_2^2}\mu_2(z) - \frac{\theta_4^2}{\theta_2^2}\mu_3(z) & -\mu_1(z) \end{pmatrix},$$

$$F(z) = \frac{1}{2}\begin{pmatrix} -\theta_2^4\left(\mu_2^2(z) + \frac{\theta_3^2}{\theta_2^2\theta_4^2}\right)\mu_1(z) & \psi_-(z)(1 - \mu_2(z)\mu_3(z)) \\ \psi_+(z)(1 + \mu_2(z)\mu_3(z)) & \theta_2^4\left(\mu_2^2(z) + \frac{\theta_3^2}{\theta_2^2\theta_4^2}\right)\mu_1(z) \end{pmatrix},$$

where $\psi_\pm = \pm\theta_4^2\mu_3(z) - \theta_3^2\mu_4(z)$ and $\mu_j(z) = \frac{1}{\theta_{j+1}(0|\tau)}\frac{\theta_{j+1}(2z|\tau)}{\theta_1(2z|\tau)}$.

Lie brackets:

$$[H(z), E(z)] = 2E(z), \quad [H(z), F(z)] = -2F(z), \quad [E(z), F(z)] = H(z).$$

Recall

$$(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2} = \bigoplus_{i=1}^{3} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}]X_i \oplus \mathbb{C}[\wp_{\frac{1}{2}\Lambda}]X_i',$$

where

$$X_i(z) = v_i \otimes w_i(z), \quad X_i'(z) = v_i \otimes w_j(z)w_k(z), \quad i = 1,2,3.$$

(Recall $\mathbb{C}\langle v_1, v_2, v_3\rangle \cong \mathfrak{sl}(2,\mathbb{C})$ and $[v_i, v_j] = \varepsilon_{ijk} v_k$.) Lie structure:

$[X_i, X_j] = \varepsilon_{ijk} X_k, \quad [X_i', X_j] = \varepsilon_{ijk} X_k \otimes w_j(z)^2, \quad [X_i', X_j'] = \varepsilon_{ijk} X_k' \otimes w_k(z)^2.$

- "complicated" brackets.

The normal form yields

$$(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2} = \mathbb{C}\langle H, E, F\rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}] \cong \mathfrak{sl}(2, \mathbb{C}[\wp_\Lambda, \wp'_\Lambda])$$

with Lie structure:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

- "easy" brackets.

Holod's Algebra

Consider

$$E_{r_1,r_2,r_3} : \lambda_i^2 - \lambda_j^2 = r_j - r_i, \quad i,j = 1,2,3$$

in $\mathbb{C}^3$. Let $\lambda = \lambda_i^2 + A_i$, where A_i are constants. Consider the complex Lie algebra with basis elements

$$X_i^{2n+2} = \lambda^n \lambda_j \lambda_k v_i, \quad X_i^{2m+1} = \lambda^m \lambda_i v_i, \quad n,m \in \mathbb{Z},$$

where i,j,k is a cyclic permutation of $1,2,3$ and v_1, v_2, v_3 are the (scaled) Pauli matrices. This is known as the Holod algebra $\mathcal{H}_\tau$. The Lie structure is given by

$$\begin{aligned}
[X_i^{2l+1}, X_j^{2s+1}] &= \varepsilon_{ijk} X_k^{2(l+s)+2}, \\
[X_i^{2l+1}, X_j^{2s}] &= \varepsilon_{ijk} (X_k^{2(l+s)+1} - A_i X_k^{2(l+s)-1}), \\
[X_i^{2l}, X_j^{2s}] &= \varepsilon_{ijk} (X_k^{2(l+s)} - A_k X_k^{2(l+s)-2}),
\end{aligned}$$

where $l,s \in \mathbb{Z}$.

- Often used in the AKS (Adler-Kostant-Symes) scheme.

Holod's Algebra

Theorem (Lombardo, Oelen ('24))

Let $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Holod's Lie algebra $\mathcal{H}_\tau$ on the complex torus $T = \mathbb{C}/\Lambda$ is the automorphic Lie algebra

$$\mathcal{H}_\tau = \left(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{T \setminus D_2 \cdot \{0, \pm z_0\}}\right)^{D_2},$$

where $\pm z_0$ are the zeros of $\wp_\Lambda$. Consequently,

$$\mathcal{H}_\tau = \mathbb{C}\langle H, E, F\rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}, \xi_{-z_0}, \xi_{z_0}] \cong \mathfrak{sl}(2, R),$$

where $H = \mathrm{Ad}(\Omega)h$, $E = \mathrm{Ad}(\Omega)e$, $F = \mathrm{Ad}(\Omega)f$, and $R = \mathcal{O}^{D_2}_{T \setminus D_2 \cdot \{0, \pm z_0\}}$.

Idea of proof

Recall the $\mathbb{C}$-basis of $\mathcal{H}_\tau$:

$$X_i^{2m+1} = \lambda^m \lambda_i v_i, \quad X_i^{2n+2} = \lambda^n \lambda_j \lambda_k v_i \quad n, m \in \mathbb{Z},$$

where $\lambda_i^2 - \lambda_j^2 = r_j - r_i$, $i, j = 1, 2, 3$ and $\lambda = \lambda_i^2 + A_i$.

- $Y_i^m(z) := \frac{1}{\wp_{\frac{1}{2}\Lambda}(z)^m} w_i(z) v_i, \quad Z_i^m(z) := \frac{1}{\wp_{\frac{1}{2}\Lambda}(z)^m} w_j(z) w_k(z) v_i$
- D_2-equivariant: e.g. $Y_i^m(\sigma(\gamma)z) = \rho(\gamma) Y_i^m(z)$ for all $\gamma \in D_2$
- $\mathcal{H}_\tau \subset \left(\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{T \setminus D_2 \cdot \{0, \pm z_0\}}\right)^{D_2} =: \mathfrak{A}(\tau, \{0, z_0, -z_0\})$
- Decomposition into subalgebras

$$\mathfrak{A}(\tau, \{0, z_0, -z_0\}) = \mathfrak{A}(\tau, \{0\}) \oplus \mathfrak{A}(\tau, \{z_0\}) \oplus \mathfrak{A}(\tau, \{-z_0\})$$

- $\mathfrak{A}(\tau, \{p\}) \subset \mathcal{H}_\tau$ for $p = 0, \pm z_0$
- N.B. for $[\tau] = [i]$, $z_0 = -z_0$
- Hence $\mathcal{H}_\tau = \mathfrak{A}(\tau, \{0, z_0, -z_0\}) \cong \mathfrak{sl}(2, \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}, \xi_{-z_0}, \xi_{z_0}])$

Uglov's Algebra

Consider the complex Lie algebra $\mathcal{E}_{k,\nu^\pm}$ generated by $\{x_i^\pm\}_{i=1,2,3}$ with the defining relations (with $\nu^\pm \in T, J_{ij} \in \mathbb{C}, w_i(z)^2 - w_j(z)^2 = J_{ij}$):

$$
\begin{aligned}
[x_i^\pm, [x_j^\pm, x_k^\pm]] &= 0, \\
[x_i^\pm, [x_i^\pm, x_k^\pm]] - [x_j^\pm, [x_j^\pm, x_k^\pm]] &= J_{ij} x_k^\pm, \\
[x_i^+, x_i^-] &= 0, \\
[x_i^\pm, x_j^\mp] &= \sqrt{-1}(w_i(\nu^\mp - \nu^\pm) x_k^\mp - w_j(\nu^\mp - \nu^\pm) x_k^\pm).
\end{aligned}
$$

Realisation as automorphic Lie algebra:

$$\mathcal{E}_{k,\nu^\pm} \cong (\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{D_2}, \quad \mathbb{T} = T \setminus D_2 \cdot \{\nu^+, \nu^-\}.$$

Uglov [Ugl93]: the Lie (bi)algebra $\mathcal{E}_{k,\nu^\pm}$ can be quantised, the corresponding quantum group is related to the eight-vertex R-matrix.

Theorem (Lombardo, Oelen ('24))

$$\mathcal{E}_{k,\nu^\pm} \cong \mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\tilde{\wp}, \tilde{\wp}', \xi],$$

where $\tilde{\wp}(z) := \wp_{\frac{1}{2}\Lambda}(z - \nu^-)$ *and* $\xi(z) := w_1(z - \nu^+) w_1(z - \nu^-)$.

Outlook

ALias $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma}$ (genus 1 case) have been classified for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and one orbit of punctures. Current research is focused on:

1. Extending the classification to more general $\mathfrak{g}$ and higher genus Riemann surfaces.
2. Establishing whether the following holds true for inner automorphisms:

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\Gamma}$$

   if $\mathbb{T} \to \mathbb{T}/\Gamma$ is unramified. In particular,

$$(\mathfrak{sl}(N, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_N \times C_N} \overset{?}{\cong} \mathfrak{sl}(N, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\wp, \wp'],$$

   where $C_N \times C_N$ acts by translations on T.
3. Applying aLias in the context of (classical/quantum) integrable systems.

Thank you for listening!

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