Jacobi forms and Kaneko–Zagier type equations

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1 Jacobi forms

Definition. Let $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$. Then *a weak Jacobi form of weight k and index m* is a holomorphic function $\varphi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following equations:

1) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \varphi(\tau, z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$;

2) $\varphi(\tau, z+\lambda\tau+\mu) = e^{-4\pi i m\lambda z - 2\pi i m \lambda^2 \tau} \varphi(\tau, z)$ for $\lambda, \mu \in \mathbb{Z}$;

3) $\varphi(\tau, z)$ has a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{n \geqslant 0} \sum_{l \in \mathbb{Z}} a(n,l) q^n \zeta^l = \sum_{n \geqslant 0} \sum_{l \in \mathbb{Z}} a(n,l) e^{2\pi i n\tau} e^{2\pi i l z}.$$

The set of all weak Jacobi forms is a bigraded ring

$$J^w_{*,*} = \bigoplus_{k,m} J^w_{k,m}.$$

Holomorphic and cusp Jacobi forms

There are two more types of Jacobi forms depending on their Fourier expansion.

A Jacobi form is a holomorphic Jacobi form $(J_{k,m})$ if

$$a(n,l) \neq 0 \Rightarrow 4nm - l^2 \geqslant 0.$$

A Jacobi form is a cusp Jacobi form $(J_{k,m}^{cusp})$ if

$$a(n,l) \neq 0 \Rightarrow 4nm - l^2 > 0.$$

It is clear that

$$J_{*,*}^{cusp} \subset J_{*,*} \subset J_{*,*}^{w}.$$

2 Examples of Jacobi forms

Let us define the odd Jacobi theta-function

$$\vartheta(\tau, z) = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n \geqslant 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n) =$$

$$= q^{\frac{1}{8}} \zeta^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(n+1)}{2}} \zeta^n.$$

It is a holomorphic Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with a multiplier system $v_\eta^3 \ltimes v_H$.

Theorem (M. Eichler, D. Zagier).

$$J_{2*,*}^w = \oplus_{k,m} J_{k,m}^w = \mathcal{M}_*[\varphi_{-2,1}, \varphi_{0,1}]$$

where

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta(\tau, z)^2}{\eta^6(\tau)} = (\zeta - 2 + \zeta^{-1}) + q \cdot (\dots) \in J_{-2,1}^w,$$

$$\varphi_{0,1}(\tau, z) = -\frac{3}{\pi^2} \wp(\tau, z) \varphi_{-2,1}(\tau, z) = (\zeta + 10 + \zeta^{-1}) + q \cdot (\dots) \in J_{0,1}^w.$$

3 The modular differential operator D_k

Let $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Then for any modular form $f = \sum a(n)q^n$ of weight k:

$$D(f) = \sum na(n)q^n.$$

Generally speaking, $D(f)$ is not a modular form, but

$$D_k(f) = 12D(f) - kE_2 \cdot f \in M_{k+2}.$$

One can show that for the Dedekind η-function $q^{\frac{1}{24}}\prod(1-q^n)$

$$\frac{D(\eta)}{\eta} = \frac{1}{24}E_2 \quad \Longrightarrow \quad D_{\frac{k}{2}}(\eta^k) = 0.$$

The well-known Ramanujan differential equations also hold

$$D_2(E_2) = -E_2^2 - E_4, \quad D_4(E_4) = -4E_6, \quad D_6(E_6) = -6E_4^2.$$

4 The modular differential operator H_k

In the case of Jacobi forms an analogue of D is **the heat operator**

$$H = \frac{3}{m}\frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right) = 12q\frac{d}{dq} - \frac{3}{m}\left(\zeta\frac{d}{d\zeta}\right)^2.$$

It transforms $\varphi_{k,m} = \sum a(n,l)q^n\zeta^l$ of weight k and index m to

$$H\left(\sum a(n,l)q^n\zeta^l\right) = \frac{3}{m}\sum(4nm - l^2)a(n,l)q^n\zeta^l.$$

As in the case of modular forms, $H(\varphi_{k,m})$ is not, generally speaking, a Jacobi form, but

$$H_k(\varphi_{k,m}) = H(\varphi_{k,m}) - \frac{(2k-1)}{2}E_2\cdot\varphi_{k,m} \in J^{w,hol,cusp}_{k+2,m}.$$

Examples. 1) $H_{\frac{1}{2}}(\vartheta) = H(\vartheta) = 0.$

$$2)\quad H_{-2}(\varphi_{-2,1}) = -3\zeta - 3\zeta^{-1} + \frac{5}{2}(\zeta - 2 + \zeta^{-1}) + q(\dots) =$$

$$= -\frac{1}{2}(\zeta + 10 + \zeta^{-1}) + q(\dots) = -\frac{1}{2}\varphi_{0,1}.$$

5 The kernel of the modular differential operator

The Leibniz product rule.

For $f \in M_{k_1}$, $\varphi \in J_{k_2,m}$:

$$H_{k_1+k_2}(f\varphi) =$$

$$= 12D(f\varphi) + \frac{3}{2\pi^2 m}\left(\frac{\partial}{\partial\mathfrak{z}}, \frac{\partial}{\partial\mathfrak{z}}\right)(f\varphi) + \left(\frac{1}{2} - k_1 - k_2\right)E_2 f\varphi =$$

$$= fH_{k_2}(\varphi) + \mathbb{D}_{k_1}(f)\varphi.$$

Remark. For any two Jacobi forms of non-zero indices the Leibniz product rule does not hold.

Proposition. If $\varphi(\tau, z)$ is a Jacobi form of weight k and index m, then $\varphi(\tau, tz)$, $t \in \mathbb{N}$ is a Jacobi form of weight k and index $t^2 m$.

Theorem (D.A., V. Gritsenko). Let $\varphi \in J^w_{k,m}$ be a weak Jacobi form of even weight and integral integral index. Then $\varphi \in \operatorname{Ker} H_k$ if and only if

$$\varphi = \varphi_{0,6}(\tau, az) \cdot \Delta(\tau)^n,$$

where $\varphi_{0,6}(\tau, az) = \frac{\vartheta(\tau,4z)}{\vartheta(\tau,2z)}$, $a \in \mathbb{N}, n \in \mathbb{Z}_{\geqslant 0}$.

6 The Kaneko–Zagier equation

The differential equation

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0$$

appeared in

M. Kaneko, D. Zagier, *Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials.* AMS/IP Stud. Adv. Math. **7** (1997), 97–126.

in connection with liftings of supersingular j-invariants of elliptic curves. In terms of D_k the Kaneko–Zagier equation could be rewritten as

$$D_{k+2}D_kf = k(k+2)E_4f.$$

One can notice that $E_4(\tau)$ and $E_6(\tau)$ satisfy this equation. There is a unique non-cusp solution for each $k \equiv 0$ or $4 \mod 6$.

7 Atkin polynomials

Theorem (O. Atkin, unpublished). There is the unique family of monic orthogonal polynomials corresponding to a unique scalar product on the space of polynomials in the modular j-invariant for which all Hecke operators are self-adjoint.

They could be defined in different ways.

1) Recursion relation:

$$A_{n+1}(j) = \left(j - 24\frac{144n^2 - 29}{(2n-1)(2n+1)} \right) A_n(j) -$$

$$-36\frac{(12n-13)(12n-7)(12n-5)(12n+1)}{n(n-1)(2n-1)^2} A_{n-1}(j)$$

2) Closed formula:

$$\sum_{i=0}^{n} 12^{3i} \left[\sum_{m=0}^{i} (-1)^m \binom{-\frac{1}{12}}{i-m} \binom{-\frac{5}{12}}{i-m} \binom{n+\frac{1}{12}}{m} \binom{n-\frac{7}{12}}{m} \binom{2n-1}{m}^{-1} \right] j^{n-i}$$

8 Kaneko–Zagier type differential equations

Kaneko–Zagier type equations are differential equations of the form

$$H_{k+2}H_k(\varphi_{k,m}) = \lambda E_4 \varphi_{k,m}.$$

In 2016 T. Kiyuna found that solutions $\varphi_{6k+4,1}$ could be written as

$$\varphi_{k,1} = \begin{cases} E_4\Delta^n P_n(j)E_{4,1} + \frac{7}{11}5E_6\Delta^n Q_n(j)E_{6,1}, & \text{if k=4+12n;} \\ E_4E_6\Delta^n R_n(j)E_{4,1} + \frac{7}{11}\Delta^{n+1}S_n(j)E_{6,1}, & \text{if k=10+12n,} \end{cases}$$

where

$$P_n(x) = (x-1728)R_{n-1}(x) + \mu_{2n-1}P_{n-1}(x),$$
$$Q_n(x) = S_{n-1}(x) + \mu_{2n-1}Q_{n-1}(x),$$
$$R_n(x) = P_n(x) + \mu_{2n}R_{n-1}(x),$$
$$S_n(x) = (x-1728)Q_n(x) + \mu_{2n}S_{n-1}(x),$$

and

$$P_0 = R_0 = S_0 = 1, Q_0 = 0, \mu_n = 48\frac{(12n-1)(12n+7)}{(4n-1)(4n+3)} \quad \text{for } k = 6n+4.$$

9 3-terms recursive relations for $\varphi_{6k+4,1}$

Polynomials $P_n(x)$, $Q_n(x)$, $R_n(x)$, $S_n(x)$ satisfy

$$P_n(x) = (x - a_n)P_{n-1}(x) - b_n P_{n-2}(x),$$

$$Q_n(x) = (x - a_n)Q_{n-1}(x) - b_n Q_{n-2}(x),$$

$$R_n(x) = (x - c_n)R_{n-1}(x) - d_n R_{n-2}(x),$$

$$S_n(x) = (x - c_n)S_{n-1}(x) - d_n S_{n-2}(x),$$

where

$$a_n = 1728 - \mu_{2n} - \mu_{2n+1} = \frac{96\left(576n^2 + 432n - 83\right)}{(8n-1)(8n+7)},$$

$$b_n = \mu_{2n-1}\mu_{2n} = \frac{2304(24n-13)(24n-5)(24n-1)(24n+7)}{(8n-5)(1-8n)^2(8n+3)},$$

$$c_n = 1728 - \mu_{2n+1} - \mu_{2n+2} = \frac{96\left(576n^2 + 1008n + 277\right)}{(8n+3)(8n+11)},$$

$$d_n = \mu_{2n}\mu_{2n+1} = \frac{2304(24n-1)(24n+7)(24n+11)(24n+19)}{(8n-1)(8n+3)^2(8n+7)}.$$

Theorem (D.A., V. Gritsenko, 2023).

Let $\varphi_{k,m} \in J^{w}_{k,m}$ be a non-trivial solution of a Kaneko–Zagier type equation. Then the weight $k \geqslant -2$;

If $k = -2$, then

$$\varphi_{k,m} = \zeta^{\pm l_1} - \zeta^{\pm l_2} + q(\ldots), \quad \text{where } \zeta^{\pm l} = \zeta^{l} + \zeta^{-l}.$$

The index $m = l_1^2 + l_2^2$ and $\lambda = \left(\frac{3l_i^2}{m} + \frac{2k-1}{2}\right)\left(\frac{3l_i^2}{m} + \frac{2k+3}{2}\right)$.

If $k > -2$, then

$$\varphi_{k,m} = \zeta^{\pm l} + q(\ldots), \quad \lambda = \left(\frac{3l^2}{m} + \frac{2k-1}{2}\right)\left(\frac{3l^2}{m} + \frac{2k+3}{2}\right).$$

Let $c_{k,l} = \frac{3l^2}{m} + \frac{2k-1}{2}$. Then $\lambda = c_{k,l}c_{k+2,l}$.

10 Rankin–Cohen brackets

For a modular form f of weight k_1, a Jacobi form φ of weight k_2 and index m, and $l \in \mathbb{Z}$ the Rankin–Cohen bracket of the type l is

$$[f,\varphi]_l = k_1 f H_{k_2}(\varphi) - c_{k_2,l} D_{k_1}(f)\varphi \in J^w_{k_1+k_2+2,m}.$$

T. Kiyuna considered $[f,\varphi] = k_1 f H_{k_2}(\varphi) - \frac{2k_2-1}{2} D_{k_1}(f)\varphi$.

Proposition. Let $\varphi_{k,m}$ be a non-trivial solution of the Kaneko–Zagier type equation with $\lambda = c_{k,l}c_{k+2,l}$. Then

$$H_{k+6}([E_4,\varphi_{k,m}]_l) = \frac{2}{3} c_{k-4,l}[E_6,\varphi_{k,m}]_l;$$

$$H_{k+8}([E_6,\varphi_{k,m}]_l) = \frac{3}{2} c_{k-6,l} E_4[E_4,\varphi_{k,m}]_l.$$

In particular, $[E_4,\varphi_{k,m}]_l$ is a solution of the Kaneko–Zagier type equation with

$$\lambda = c_{k-4,l}c_{k-6,l}.$$

11 Main result

Lemma. Let $\Phi_{k,m} = \zeta^{\pm l} + q(\ldots) \in J^w_{k,m}$ is a solution of a Kaneko–Zagier equation, $k \geqslant 0$. Then

$$\Phi_{k+6,m} = E_6\Phi_{k,m} - \frac{1}{8c_{k+1,l}}[E_4, \Phi_{k,m}]_l = \zeta^{\pm l} + q(\ldots) \in J^w_{k+6,m}$$

is a solution of a Kaneko–Zagier equation.

Lemma. Let
$\Phi_{k,m} = \zeta^{\pm l} + q(\ldots) \in J^w_{k,m}$ and $\Phi_{k-6,m} = \zeta^{\pm l} + q(\ldots) \in J^w_{k-6,m}$ are solutions of a Kaneko–Zagier equations, $k-6 \geqslant 0$. Then

$$\Phi_{k+6,m} = E_6\Phi_{k,m} + 432\frac{c_{k,l}c_{k-4,l}}{c_{k+1,l}c_{k-5,l}}\Delta\Phi_{k-6,m} =$$

$$= \zeta^{\pm l} + q(\ldots) \in J^w_{k+6,m}$$

is a solution of a Kaneko–Zagier equation.

Theorem. Let $f_k \in \mathcal{M}_k$ is a solution of Kaneko–Zagier equation. Then $f_k \varphi_{0,6}$ is a solution of Kaneko–Zagier type equation. There are also 4 special sequences of non-trivial solutions of Kaneko–Zagier type equations.

Solutions of index 1:

$$\begin{tikzcd}
 & \varphi_{4,1} \arrow[r] & \varphi_{10,1} \arrow[r] & \varphi_{16,1} \arrow[r] & \ldots \arrow[r] & \varphi_{6k+4,1} \arrow[r] & \ldots \\
\varphi_{-2,1} \arrow[ur] \arrow[dr] & & & & & & \\
 & \psi_{4,1} \arrow[r] & \psi_{10,1} \arrow[r] & \psi_{16,1} \arrow[r] & \ldots \arrow[r] & \psi_{6k+4,1} \arrow[r] & \ldots
\end{tikzcd}$$

Here

$$\varphi_{4,1} = -\frac{1}{24}[E_4, \varphi_{-2,1}]_1 = 1 + q(\ldots),$$

$$\varphi_{10,1} = E_6 \varphi_{4,1} - 56 \Delta \varphi_{-2,1} = 1 + q(\ldots),$$

and

$$\psi_{4,1} = -\frac{1}{12}[E_4, \varphi_{-2,1}]_0 = \zeta^{\pm 1} + q(\ldots),$$

$$\psi_{10,1} = E_6 \psi_{4,1} + 624 \Delta \varphi_{-2,1} = \zeta^{\pm 1} + q(\ldots).$$

Solutions of index 10:

$$\begin{tikzcd}[column sep=small]
 & \Phi_{4,10} \arrow[r] & \Phi_{10,10} \arrow[r] & \Phi_{16,10} \arrow[r] & \ldots \arrow[r] & \Phi_{6k+4,10} \arrow[r] & \ldots \\
\Phi_{-2,10} \arrow[ur] \arrow[dr] & & & & & & \\
 & \Psi_{4,10} \arrow[r] & \Psi_{10,10} \arrow[r] & \Psi_{16,10} \arrow[r] & \ldots \arrow[r] & \Psi_{6k+4,10} \arrow[r] & \ldots
\end{tikzcd}$$

Here

$$\Phi_{-2,10} = \frac{\vartheta(\tau,2z)\vartheta(\tau,4z)}{\eta^6(\tau)} = \zeta^{\pm 3} - \zeta^{\pm 1} + q(\ldots),$$

$$\Phi_{4,10} = -\frac{5}{48}[E_4, \Phi_{-2,1}]_3 = \zeta^{\pm 1} + q(\ldots),$$

$$\Phi_{10,10} = E_6\Phi_{4,10} - 57\Delta\Phi_{-2,10} = \zeta^{\pm 1} + q^{(}\ldots),$$

and

$$\Psi_{4,10} = -\frac{5}{48}[E_4, \varphi_{-2,10}]_1 = \zeta^{\pm 3} + q(\ldots),$$

$$\Psi_{10,10} = E_6\Psi_{4,10} + 682\Delta\Phi_{-2,10} = \zeta^{\pm 3} + q(\ldots).$$

12 Idea of the proof.

Negative weight. Any Jacobi form of weight -2 is divisible by $\varphi_{-2,1}$. Let

$$\phi_{-2,m} = \varphi_{-2,1}\phi_{0,m-1}.$$

Lemma (Gritsenko, '99). For any weak Jacobi form $\phi_{0,m} = \sum_{n,l} a(n,l)q^n\zeta^l$ the following identities hold

$$m\sum_l a(0,l) = 6\sum_l l^2 a(0,l), \qquad 24m\sum_l a(0,l) = \sum_n (m-6n^2)a(1,n).$$

Non-negative weight. Let $\phi_{k,m} = \zeta^{\pm l} + q(\dots)$ is a solution of a Kaneko–Zagier type equation.

Lemma. Then $[E_4, \phi_{k,m}]_l = \Delta\phi_{k-6,m}$, where $\phi_{k-6,m} = \zeta^{\pm l} + q(\dots)$ is also a solution.

Lemma. If $[E_4, \phi_{k,m}]_l = 0$, then $\phi_{k,m} = \varphi_{0,6}$ or $E_4\varphi_{0,6}$.

13 Recursive relations for $\varphi_{6k+4,1}$

T. Kiyuna found that solutions $\varphi_{6k+4,1}$ could be written as

$$\varphi_{k,1} = \begin{cases} \frac{1}{12}E_4\Delta^n P_n(j)\varphi_{0,1} - \frac{1}{12}E_6\Delta^n Q_n(j)\varphi_{-2,1}, & \text{if k=4+12n;} \\ \frac{1}{12}E_4E_6\Delta^n R_n(j)\varphi_{0,1} - \frac{1}{12}\Delta^{n+1}S_n(j)\varphi_{-2,1}, & \text{if k=10+12n,} \end{cases}$$

where

$$P_n(x) = (x-1728)R_{n-1}(x) + \mu_{2n-1}P_{n-1}(x),$$

$$Q_n(x) = S_{n-1}(x) + \mu_{2n-1}Q_{n-1}(x),$$

$$R_n(x) = P_n(x) + \mu_{2n}R_{n-1}(x),$$

$$S_n(x) = (x-1728)Q_n(x) + \mu_{2n}S_{n-1}(x),$$

and

$$P_0 = Q_0 = R_0 = 1,$$

$$S_0 = x - 1056,$$

$$\mu_n = 432\frac{c_{k,0}c_{k-4,0}}{c_{k+1,0}c_{k-5,0}} \qquad \text{for } k = 6n+4.$$

14 3-terms recursive relations for $\varphi_{6k+4,1}$

Polynomials $P_n(x)$, $Q_n(x)$, $R_n(x)$, $S_n(x)$ satisfy

$$P_n(x) = (x - a_n)P_{n-1}(x) - b_n P_{n-2}(x),$$

$$Q_n(x) = (x - a_n)Q_{n-1}(x) - b_n Q_{n-2}(x),$$

$$R_n(x) = (x - c_n)R_{n-1}(x) - d_n R_{n-2}(x),$$

$$S_n(x) = (x - c_n)S_{n-1}(x) - d_n S_{n-2}(x),$$

where

$$a_n = 1728 - \mu_{2n} - \mu_{2n+1} = \frac{96\left(576n^2 + 432n - 83\right)}{(8n-1)(8n+7)},$$

$$b_n = \mu_{2n-1}\mu_{2n} = \frac{2304(24n-13)(24n-5)(24n-1)(24n+7)}{(8n-5)(1-8n)^2(8n+3)},$$

$$c_n = 1728 - \mu_{2n+1} - \mu_{2n+2} = \frac{96\left(576n^2 + 1008n + 277\right)}{(8n+3)(8n+11)},$$

$$d_n = \mu_{2n}\mu_{2n+1} = \frac{2304(24n-1)(24n+7)(24n+11)(24n+19)}{(8n-1)(8n+3)^2(8n+7)}.$$

15 Recursive relations for $\psi_{6k+4,1}$

Solutions $\psi_{6k+4,1}$ could be written as

$$\psi_{k,1} = \begin{cases} E_4\Delta^n P_n(j)\varphi_{0,1} + 5E_6\Delta^n Q_n(j)\varphi_{-2,1}, & \text{if k=4+12n;} \\ E_4E_6\Delta^n R_n(j)\varphi_{0,1} + 5\Delta^{n+1} S_n(j)\varphi_{-2,1}, & \text{if k=10+12n,} \end{cases}$$

where

$$P_n(x) = (x-1728)R_{n-1}(x) + \mu_{2n-1}P_{n-1}(x),$$
$$Q_n(x) = S_{n-1}(x) + \mu_{2n-1}Q_{n-1}(x),$$
$$R_n(x) = P_n(x) + \mu_{2n}R_{n-1}(x),$$
$$S_n(x) = (x-1728)Q_n(x) + \mu_{2n}S_{n-1}(x),$$

and

$$P_0 = Q_0 = R_0 = 1,$$
$$S_0 = x - \frac{4896}{5},$$
$$\mu_n = 432\frac{c_{k,1}c_{k-4,1}}{c_{k+1,1}c_{k-5,1}} \qquad \text{for } k = 6n+4.$$

16 3-terms recursive relations for $\psi_{6k+4,1}$

Polynomials $P_n(x)$, $Q_n(x)$, $R_n(x)$, $S_n(x)$ satisfy

$$P_n(x) = (x - a_n)P_{n-1}(x) - b_n P_{n-2}(x),$$

$$Q_n(x) = (x - a_n)Q_{n-1}(x) - b_n Q_{n-2}(x),$$

$$R_n(x) = (x - c_n)R_{n-1}(x) - d_n R_{n-2}(x),$$

$$S_n(x) = (x - c_n)S_{n-1}(x) - d_n S_{n-2}(x),$$

where

$$a_n = 1728 - \mu_{2n} - \mu_{2n+1} = \frac{96(576n^2 - 432n - 83)}{(8n-7)(8n+1)},$$

$$b_n = \mu_{2n-1}\mu_{2n} = \frac{2304(24n-13)(24n-5)(24n-1)(24n+7)}{(8n-5)(1-8n)^2(8n+3)},$$

$$c_n = 1728 - \mu_{2n+1} - \mu_{2n+2} = \frac{96(576n^2 + 144n - 155)}{(8n-3)(8n+5)},$$

$$d_n = \mu_{2n}\mu_{2n+1} = \frac{2304(24n-19)(24n-11)(24n-7)(24n+1)}{(8n-7)(8n-3)^2(8n+1)}.$$

17 Atkin-like polynomials

Introduced by J. Wimp ('87) and studied by A. El-Guindy and M.E.H. Ismail ('14), Atkin-like polynomials could be defined by closed formulae

$$G_n(x;a,b,c) = (-1)^n \frac{(c+1)_n(b+c+1)_n}{(a+b+2c+n+1)_n n!} \times \sum_{k=0}^{n} \frac{(-n)_k(n+a+b+2c+1)_k}{(c+1)_k(b+c+1)_k} x^k \times {}_4F_3\left(\left.\begin{matrix} k-n,\ n+k+a+b+2c+1,\ b+c,\ c \\ k+b+c+1,\ k+c+1,\ a+b+2c \end{matrix}\right| 1\right)$$

$$H_n(x;a,b,c) = (-1)^n \frac{(c+1)_n(b+c+1)_n}{(a+b+2c+n+1)_n n!} \times \sum_{k=0}^{n} \frac{(-n)_k(n+a+b+2c+1)_k}{(c+1)_k(b+c+1)_k} x^k \times {}_4F_3\left(\left.\begin{matrix} k-n,\ n+k+a+b+2c+1,\ b+c+1,\ c \\ k+b+c+1,\ k+c+1,\ a+b+2c+1 \end{matrix}\right| 1\right)$$

18 Expression of solutions in Atkin-like polynomials

Let $y = 1 - \frac{x}{1728}$. For solutions $\varphi_{6k+4,1}$

$$P_n(x) = -\frac{7}{65} G_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{13}{24}\right) + \frac{72}{65} H_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{13}{24}\right);$$

$$Q_n(x) = \frac{7}{13} G_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{13}{24}\right) + \frac{6}{13} H_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{13}{24}\right);$$

$$R_n(x) = H_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{1}{24}\right);$$

$$S_n(x) = -16720 G_{n+1}\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{1}{24}\right) - \frac{347}{85} G_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{1}{24}\right) +$$

$$+ \frac{432}{85} H_n\left(y; \frac{1}{3}, \frac{1}{2}, -\frac{1}{24}\right).$$

For solutions $\psi_{6k+4,1}$

$$P_n(x) = -\frac{65}{7}G_n\left(y;\frac{1}{3},\frac{1}{2},-\frac{7}{24}\right) + \frac{72}{7}H_n\left(y;\frac{1}{3},\frac{1}{2},-\frac{7}{24}\right);$$

$$Q_n(x) = \frac{13}{7}G_n\left(y;\frac{1}{3},\frac{1}{2},-\frac{7}{24}\right) - \frac{6}{7}H_n\left(y;\frac{1}{3},\frac{1}{2},-\frac{7}{24}\right);$$

$$R_n(x) = H_n\left(y;\frac{1}{3},\frac{1}{2},-\frac{1}{24}\right);$$

$$S_n(x) = -\frac{4021}{16720}G_{n+1}\left(y;\frac{1}{3},\frac{1}{2},\frac{5}{24}\right) + 816G_n\left(y;\frac{1}{3},\frac{1}{2},\frac{5}{24}\right) + \\ + \frac{20741}{16720}H_{n+1}\left(y;\frac{1}{3},\frac{1}{2},\frac{5}{24}\right) + H_n\left(y;\frac{1}{3},\frac{1}{2},\frac{5}{24}\right).$$

Thank you!