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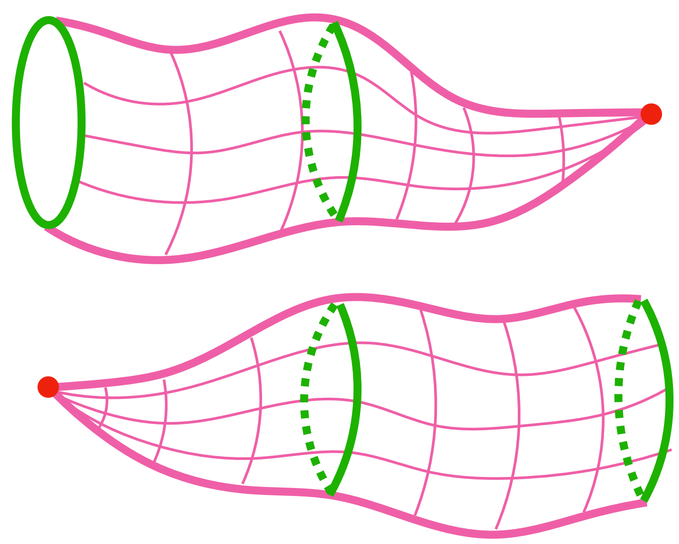
Periods of fibre products of elliptic surfaces

arxiv:2505.07685



Periods of algebraic varieties

A **period** of an algebraic variety is the integral of a rational form of the variety on a cycle.

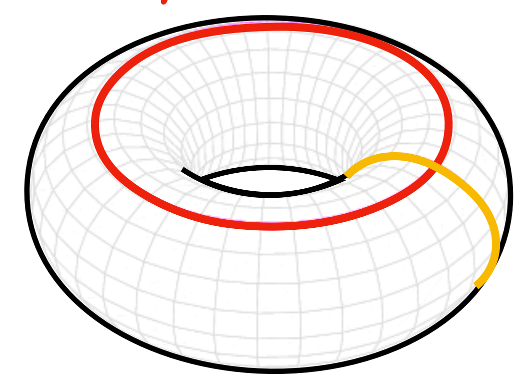
Some integration domain without boundary

$$\int_{T(\gamma)} \frac{A}{P^k}$$

A is a polynomial

P defines a smooth variety $V(P) = \{z \mid P(z) = 0\}$

γ



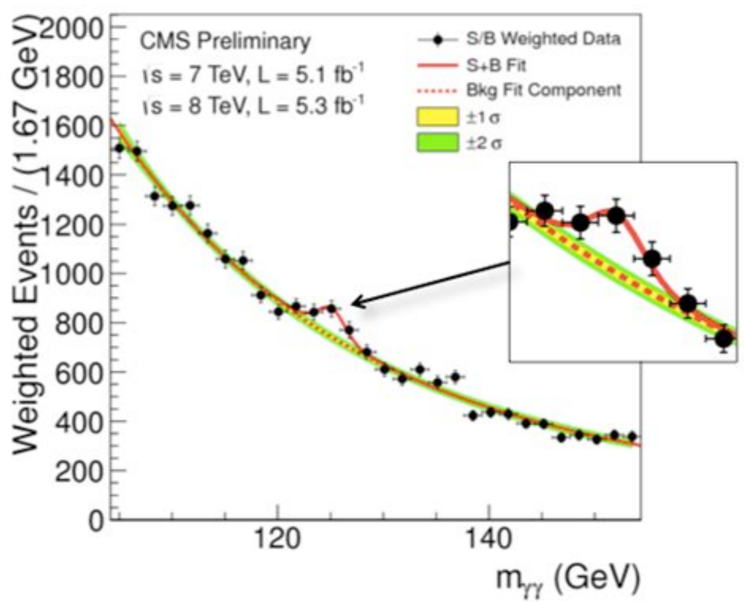
An elliptic curve

Torelli-type theorem for K3 surfaces:
Two K3 surfaces are isomorphic if and only if they have “the same” periods.

They describe the comparison between **topological data** (cycles) and **algebraic data** (algebraic De Rham forms).

$$H_n(S, \mathbb{Z}) \times H^n_{DR}(S) \to \mathbb{C} \qquad \gamma, \omega \mapsto \int_\gamma \omega$$

Motivation and goals



Periods appear in diverse fields of mathematics and physics, such as **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

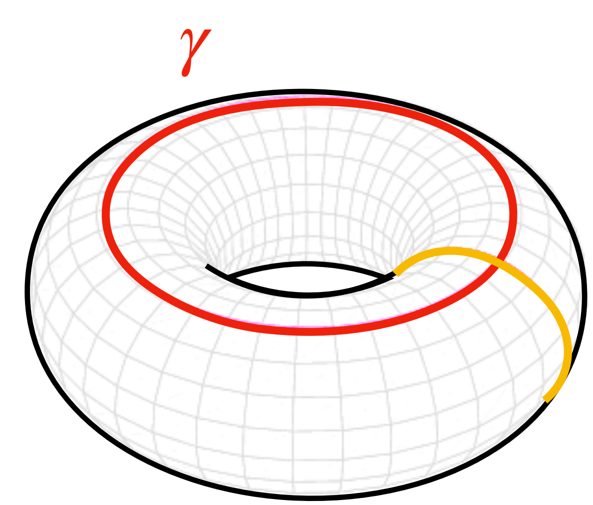
Hundreds of digits

Sufficiently many to recover algebraic invariants

Goal: compute numerical approximations of these integrals with **large precision.**

For this, we need an appropriate description of the integrals.

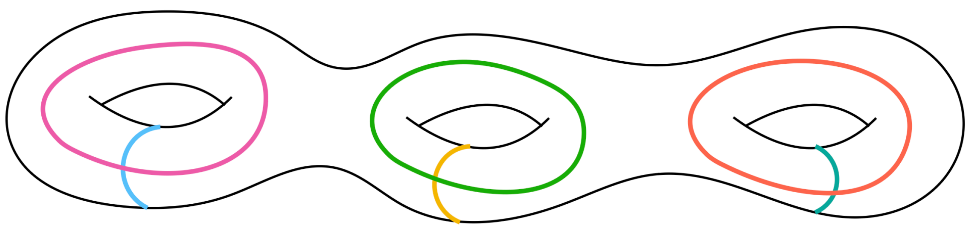
In particular we will focus on **understanding the cycles of integration** (the homology), how to represent them in a way that makes integration concrete, and how to compute a basis of them.



Furthermore we want this to be **effective** and **efficient**.

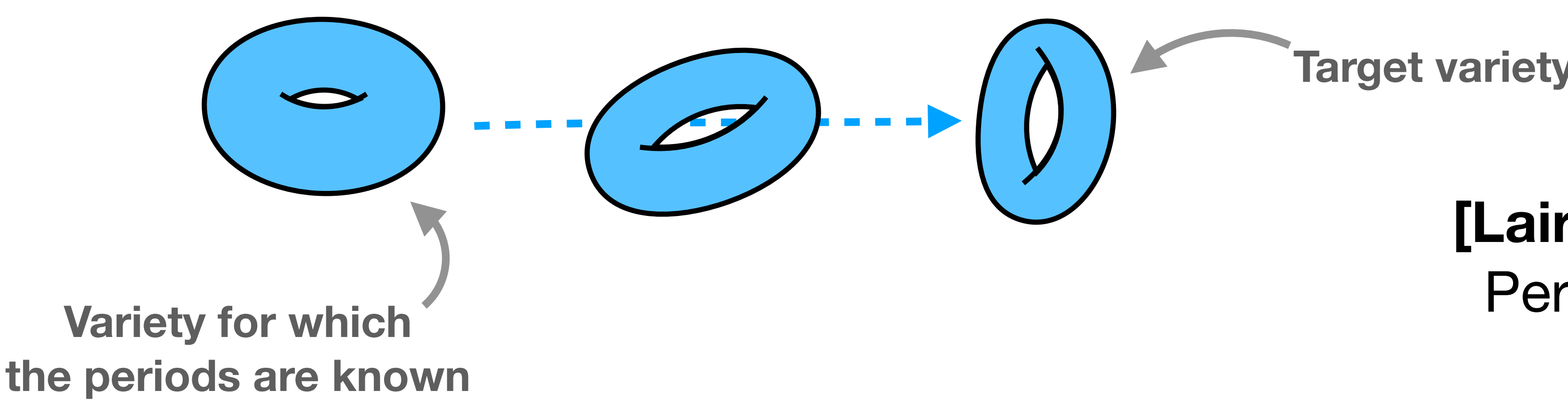
Previous works on period computations

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]: Algebraic curves (Riemann surfaces)



[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]: Higher dimensional varieties (double covers of $\mathbb{P}^2/\mathbb{P}^3$ ramified along a hyperplane arrangement)

[Sertöz 2019]: compute the period matrix of smooth projective hypersurfaces by **deformation**.



[Lairez PP Vanhove 2025]: Periods of hypersurfaces

[PP 2025 x2]: Periods of elliptic surfaces and fibre products of elliptic surfaces

[Đonlagić 2025]: Periods of fibre products of elliptic surfaces with semi-simple singular fibres

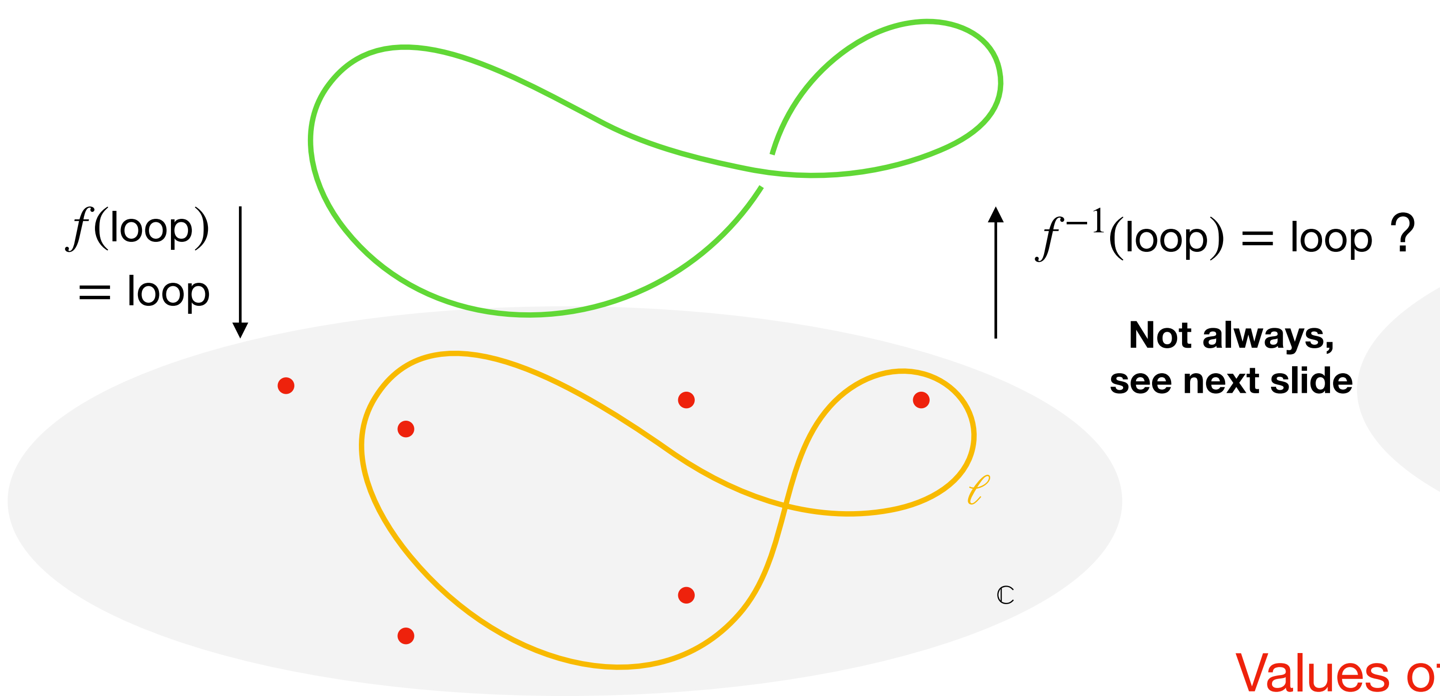
Periods of algebraic curves

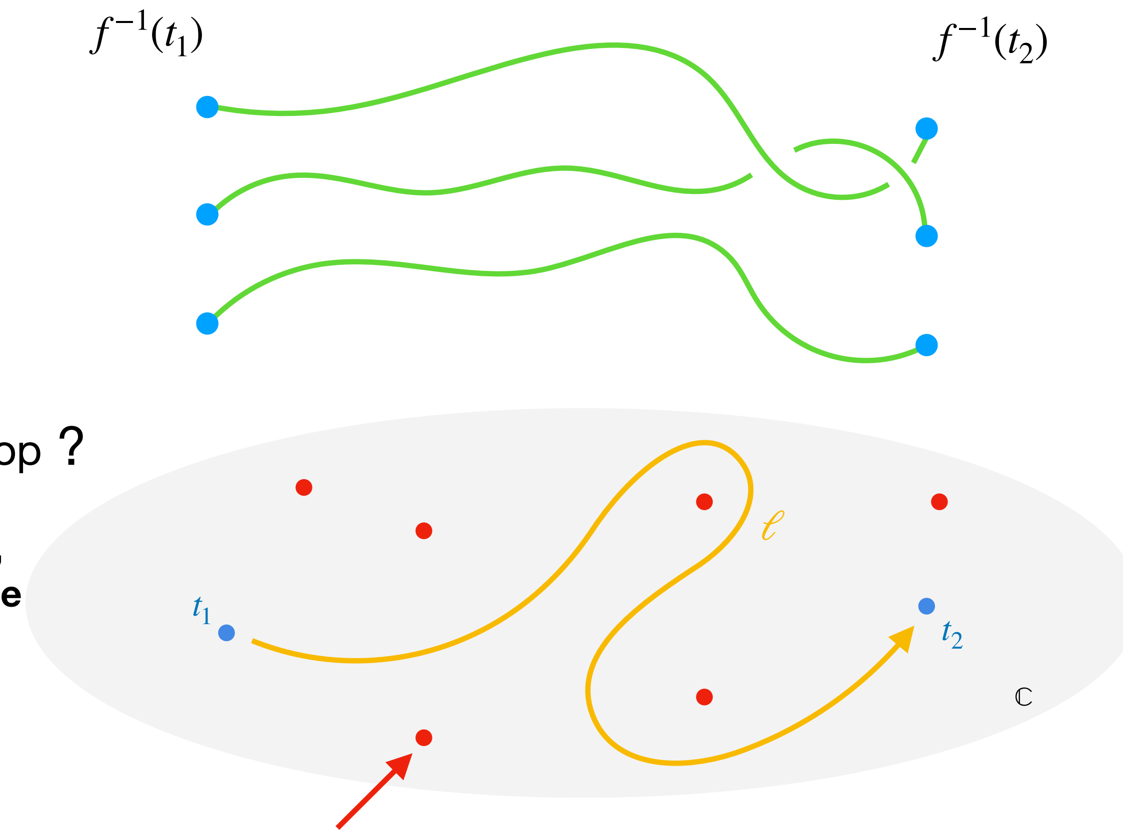
First example: algebraic curves

Let $\mathscr{X}$ be the elliptic curve defined by
$P = y^3 + x^3 + 1 = 0$
and let $f : (x, y) \mapsto y/(2x + 1)$ be a generic projection.

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x+1)) \mid P\left(x, t(2x+1)\right) = 0\}$.
It deforms continuously with respect to t.

In dimension 1, we are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).

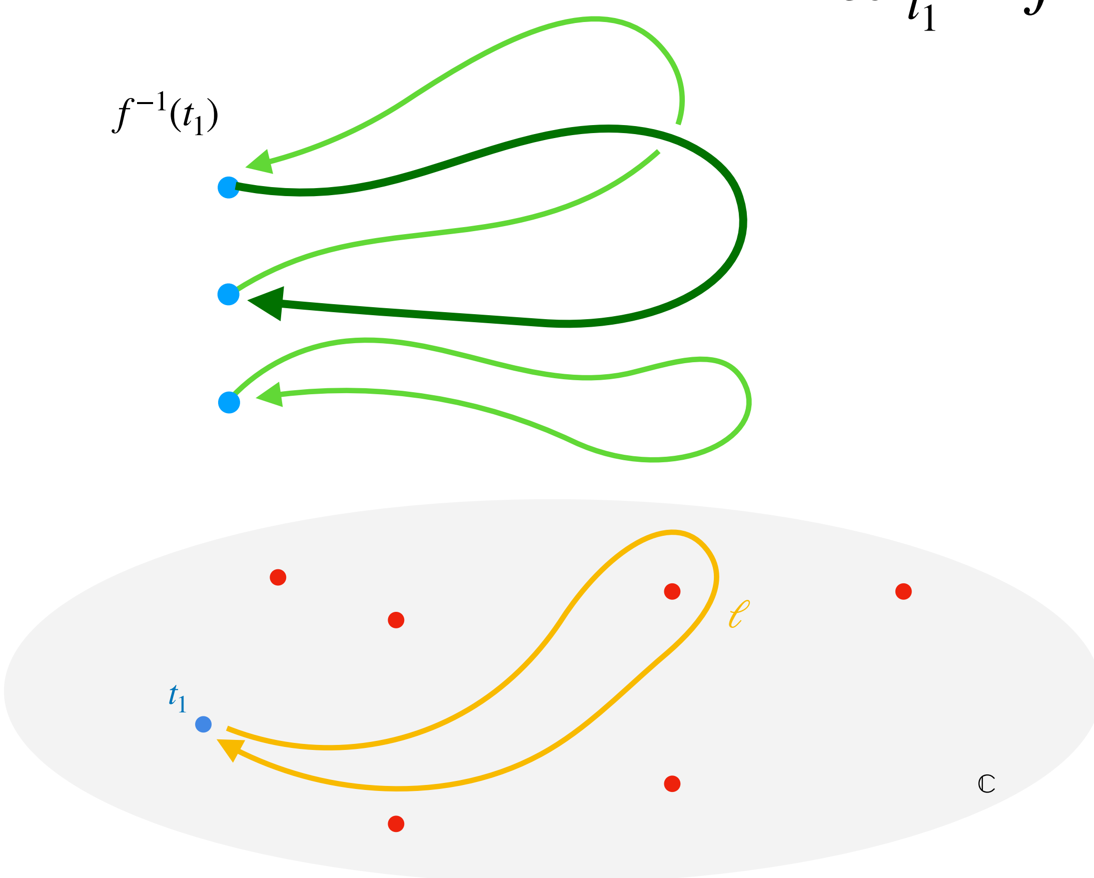


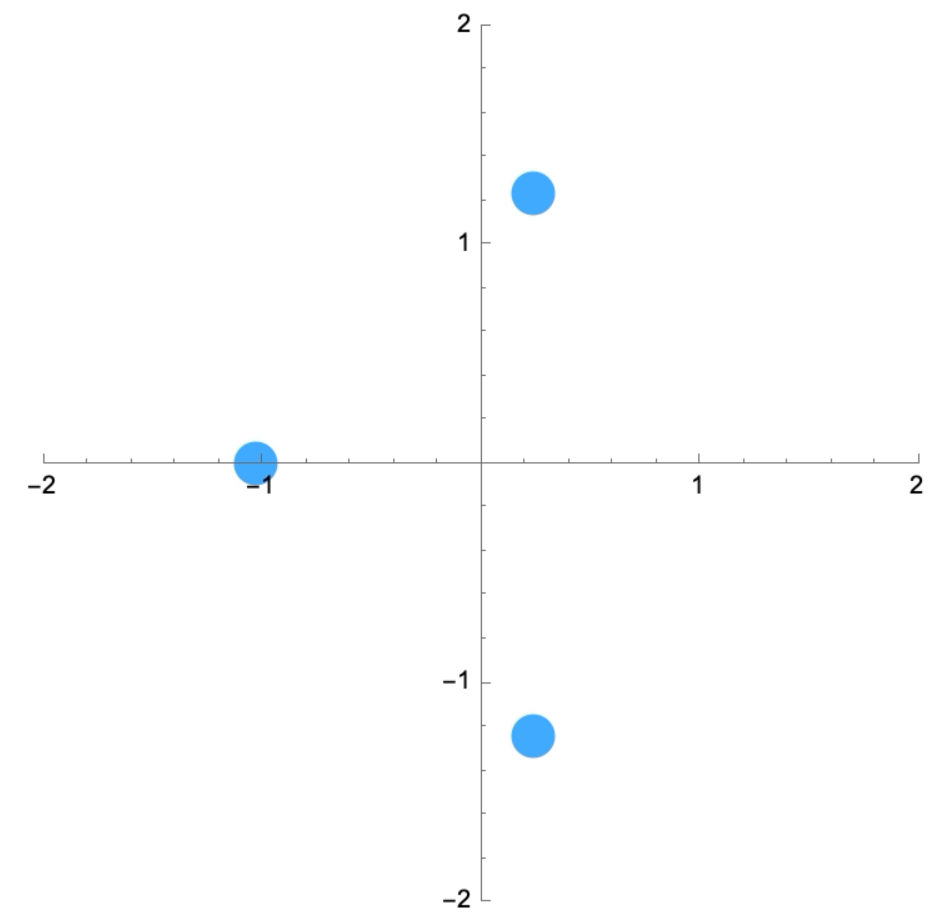


Values of t for which
$P(x, t(2x+1)) = t^3(2x+1)^3 + x^3 + 1$
has a double root (critical values)

What happens when you loop around a critical point?

A loop ℓ in $\mathbb{C}$ pointed at t_1 induces a permutation of
$$\mathcal{X}_{t_1} = f^{-1}(t_1).$$





This permutation is called the **action of monodromy along ℓ** on $\mathcal{X}_{t_1}$.

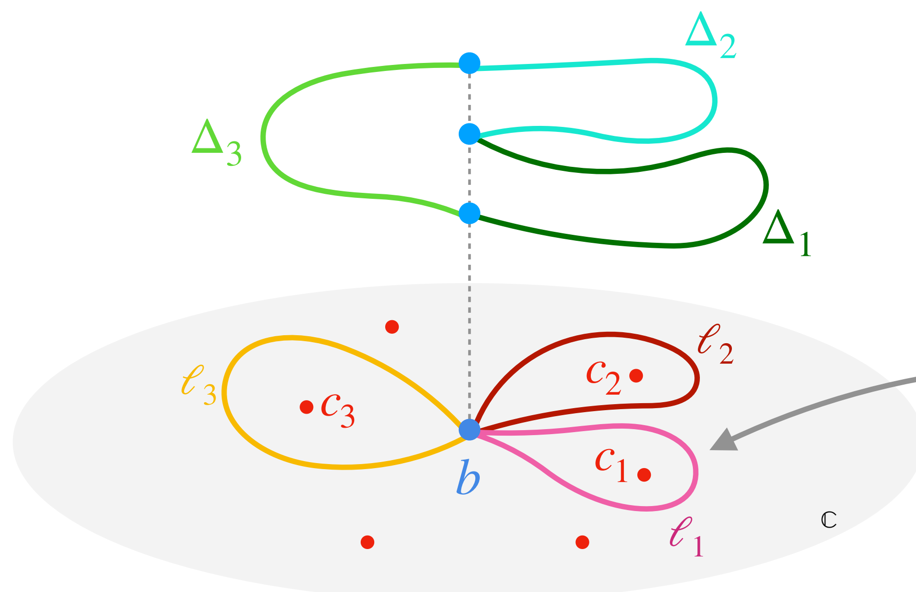
It is denoted ℓ_*.

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

Periods of algebraic curves

The lift of a simple loop ℓ around a critical value c that has a non-trivial boundary in $\mathscr{X}_b$ is called the **thimble** of c. It is an element of $H_1(\mathscr{X}, \mathscr{X}_b)$.

Relative homology of the pair $(\mathscr{X}, \mathscr{X}_b)$



Simple loop around c_1

Thimbles serve as building blocks to recover $H_1(\mathscr{X})$.
It is sufficient to glue thimbles together in a way such that their boundaries cancels.

Concretely, we take the kernel of the boundary map

$$\delta : H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

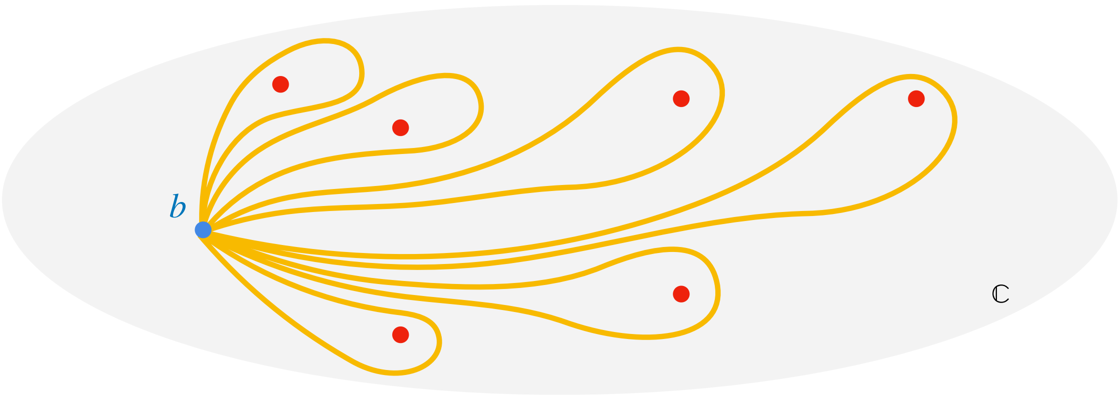
Fact: all of $H_1(\mathscr{X})$ can be recovered this way.

$$0 \to H_1(\mathscr{X}) \to H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$$

Generated by thimbles

Computing periods of algebraic curves

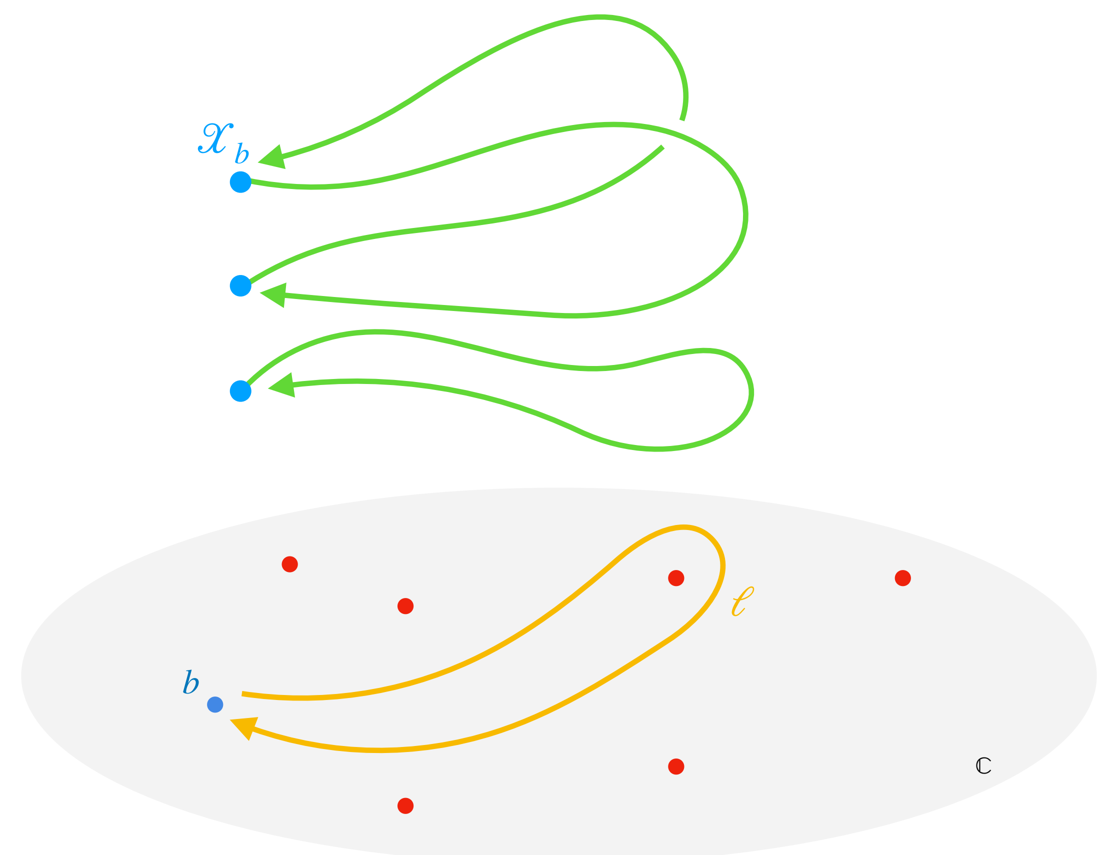
1. Compute simple loops $\ell_1, \ldots, \ell_{\text{\#crit.}}$ around the critical values —
basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. val.}\})$



Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. val.}\})$

2. For each i compute the action of monodromy along ℓ_i on $\mathcal{X}_b$ (transposition)



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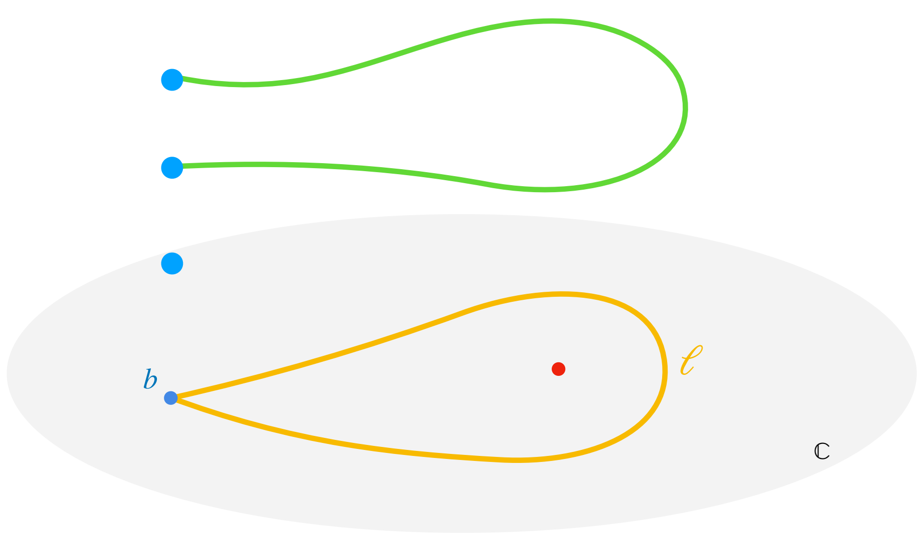
3. This provides the corresponding thimble Δ_i. Its boundary is the difference of the two points of $\mathscr{X}_b$ that are permuted.



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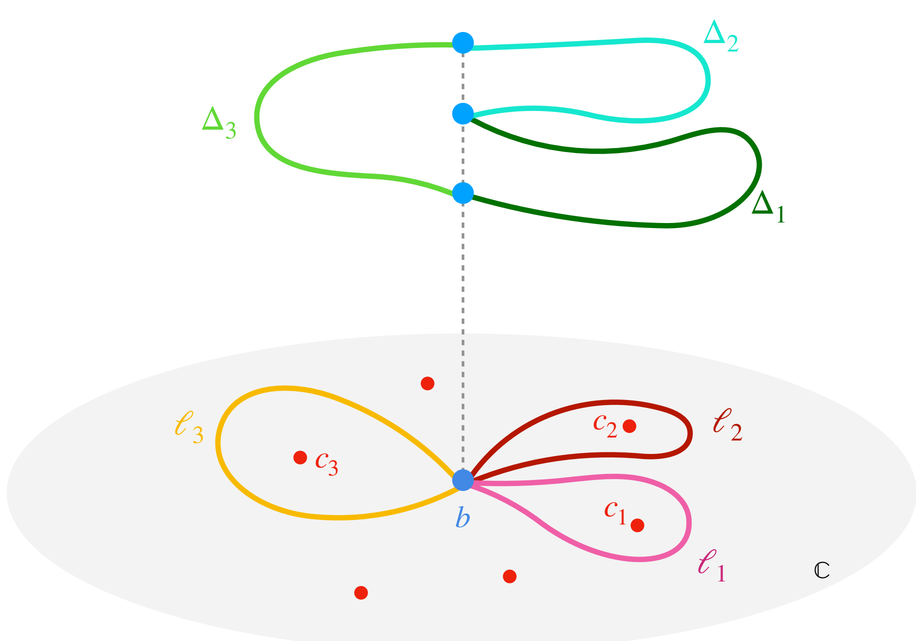
4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$



Computing periods of algebraic curves

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4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$

5. Periods are integrals along these loops → we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

Elliptic surfaces

[Lairez PP Vanhove 2024]

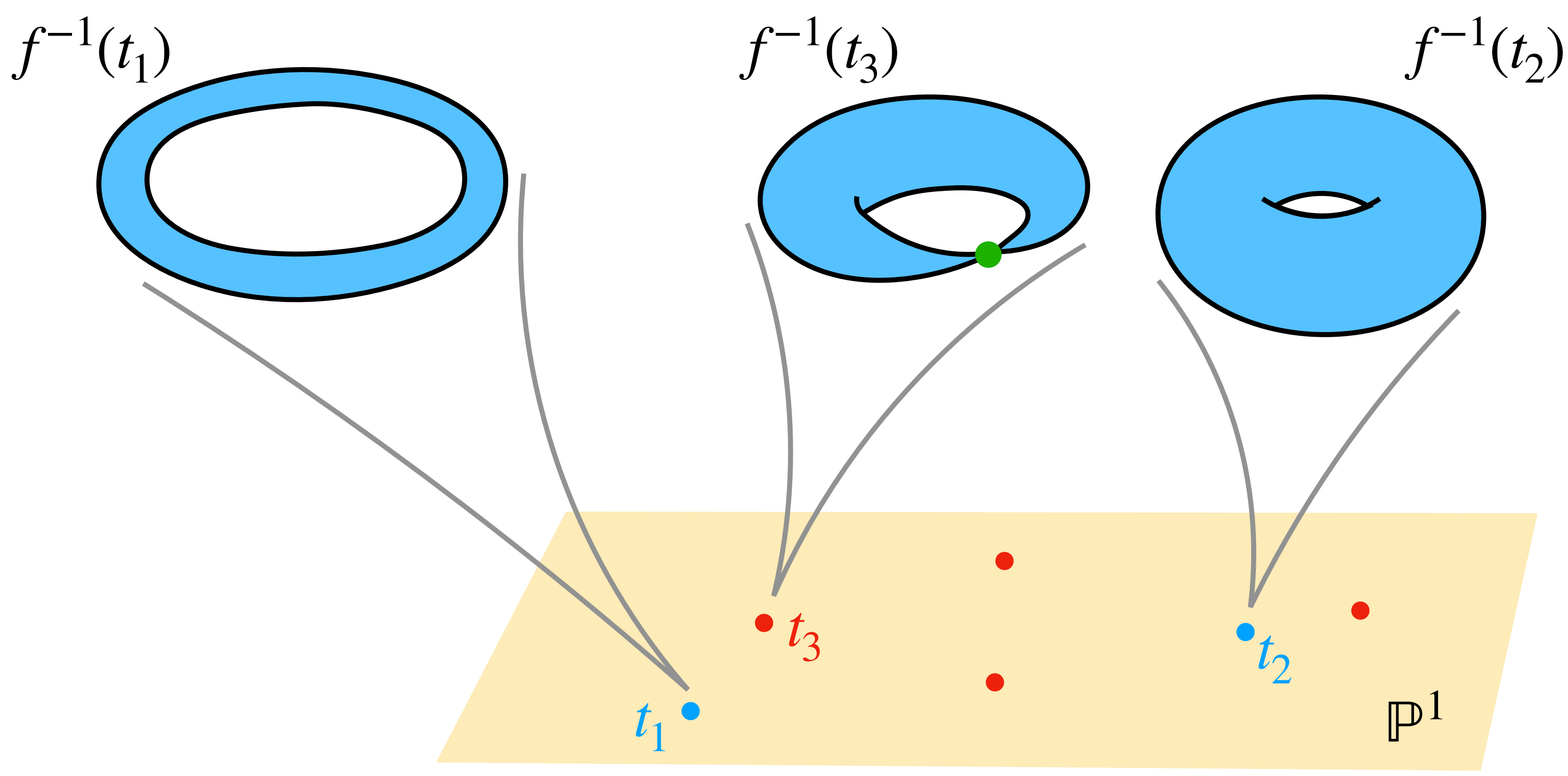
[PP 2024]

Elliptic surfaces

An **elliptic surface** S is a smooth algebraic surface equipped with a map to the projective line

$$f : S \to \mathbb{P}^1$$

such that all but finitely many fibres $f^{-1}(t)$ are elliptic curves.



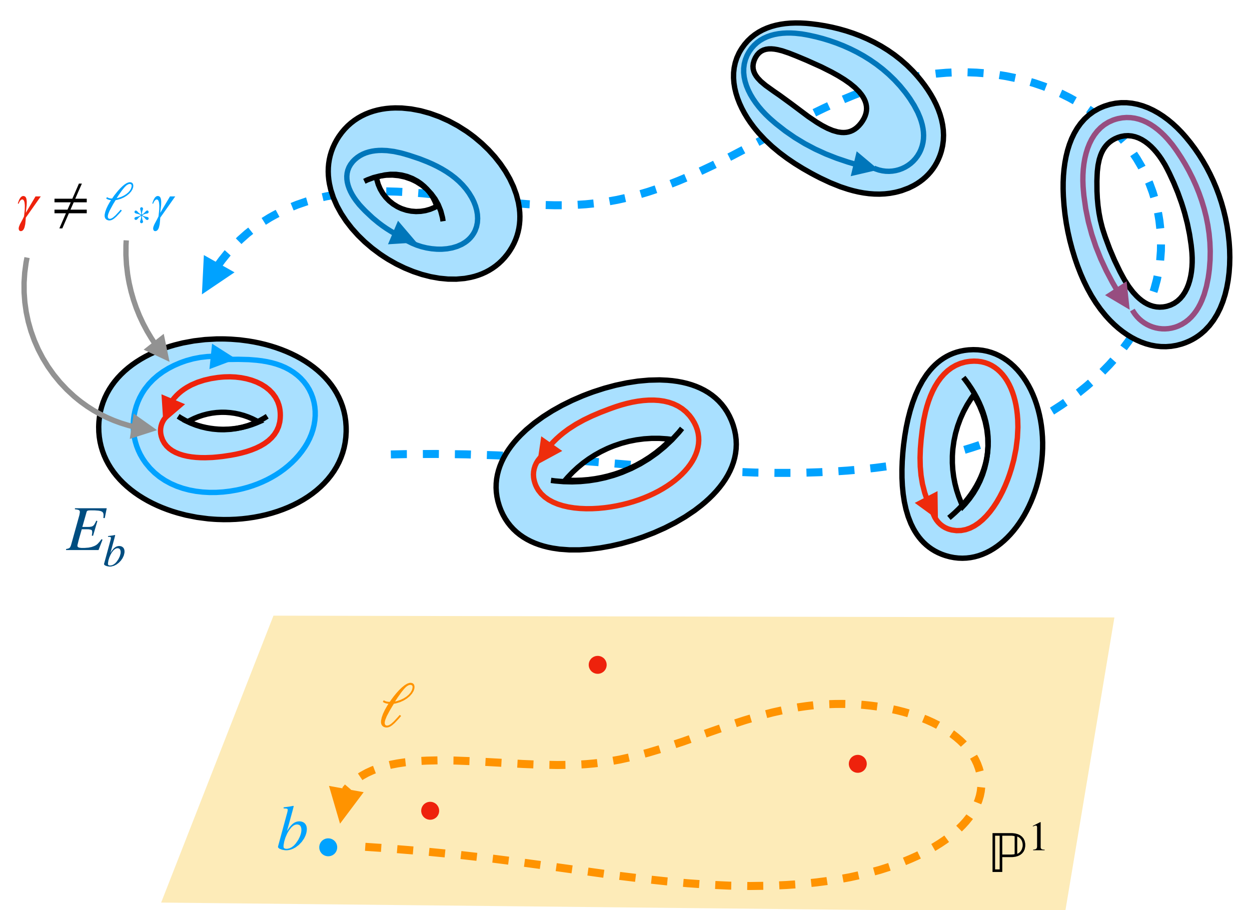
We will assume the surface has a **section**.

Monodromy

Let $\mathscr{X}$ be a smooth surface in $\mathbb{P}^3$. We consider a projection $\mathscr{X} \to \mathbb{P}^1$.
The fibre $\mathscr{X}_t = f^{-1}(t)$ is a curve, which deforms continuously as t moves in $\mathbb{P}^1$.

Ehresmann's fibration theorem

The map $\ell_* : H_1(\mathscr{X}_b) \to H_1(\mathscr{X}_b)$ induced by this deformation along a loop ℓ is called the **monodromy along ℓ**.



In other words, the first homology of the fibre is locally constant.
The associated sheaf over $\mathbb{C}\backslash\Sigma$ defines a **local system**.

Extensions

We can recover 2-cycles for the periods of elliptic surfaces as **extensions** of 1-cycles of the fibre.

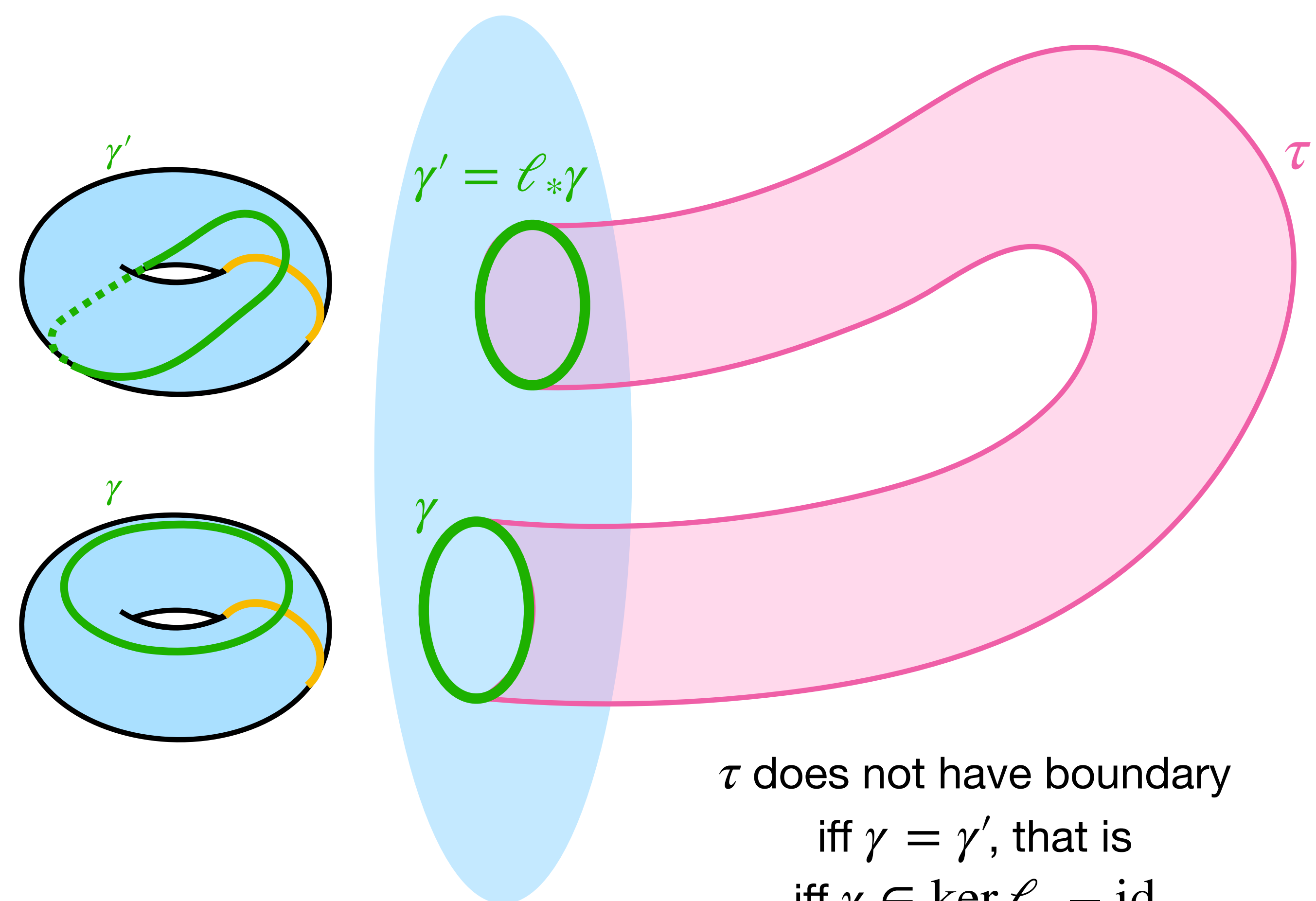
$$\pi_1(\mathbb{P}^1 \backslash \Sigma, b) \times H_1(E_b) \to H_2(S, E_b)$$
$$\ell, \gamma \mapsto \tau_\ell(\gamma)$$

This description of cycles is well-suited for integrating the periods:

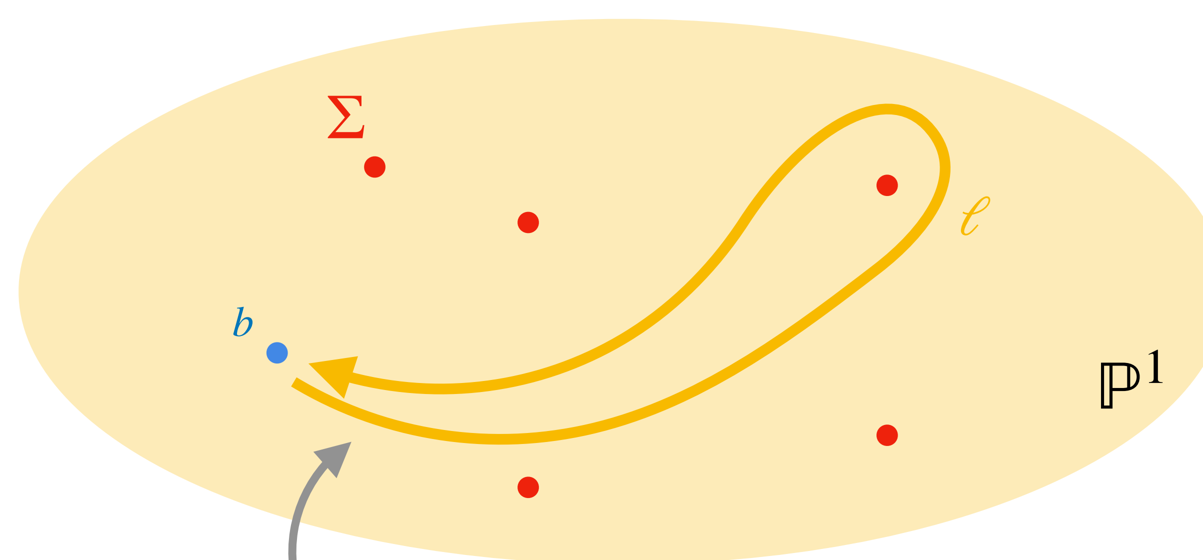
$$\int_{\tau_\ell(\gamma)} f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_\gamma f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Two line integrals:
we know how to compute these efficiently!
[Chudnovsky[2], Van der Hoeven, Mezzarobba]

$$\partial\tau_\ell(\gamma) = \gamma' - \gamma$$



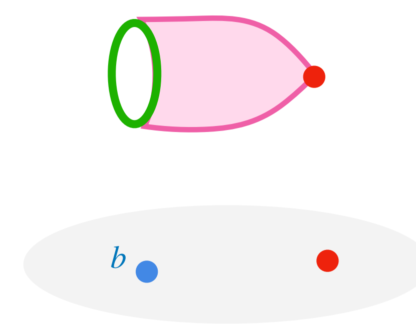
τ does not have boundary
iff $\gamma = \gamma'$, that is
iff $\gamma \in \ker \ell_* - \mathrm{id}$

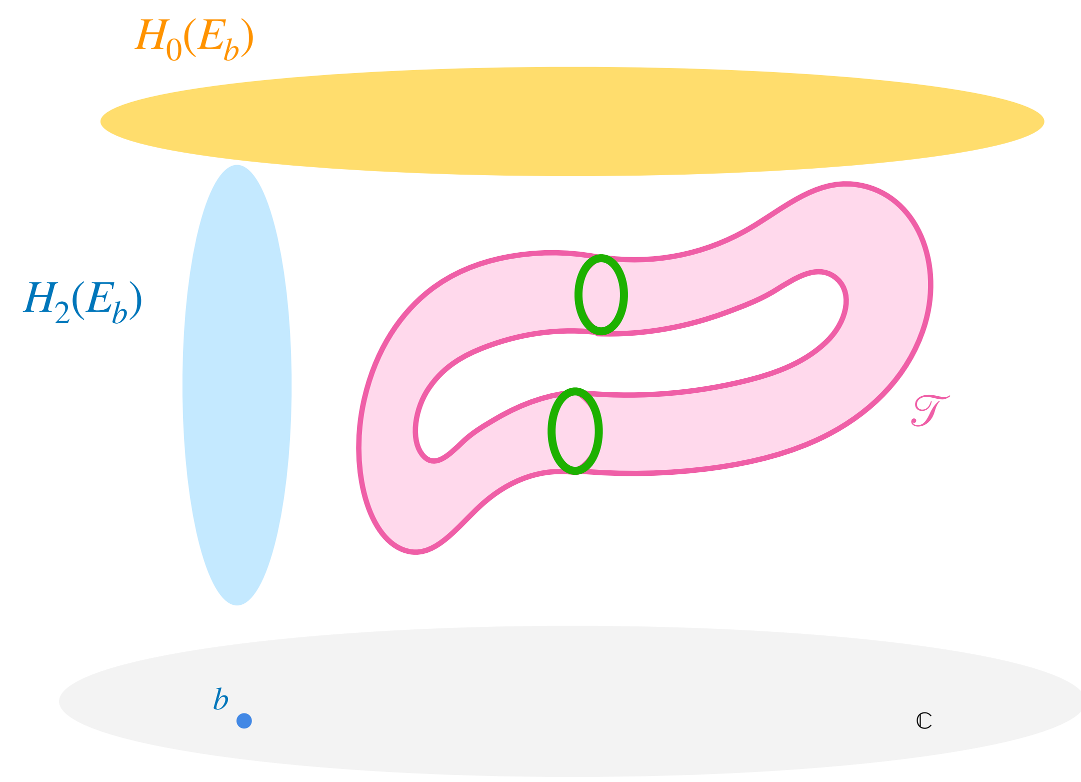


such a path is called a **simple loop**

Parabolic homology

Each **simple loop** ℓ contributes relative homology classes, called **thimbles**, in $H_2(S, E_b)$.





Thimbles serve as building blocks for extensions: we can glue thimbles together in a way that matches their **boundary** to obtain closed cycles.

Obtained from the monodromy :
$\partial\tau_\ell(\gamma) = \ell_*\gamma - \gamma$

Furthermore $H_2(S)$ is generated by **extensions**, **fibre components**, and a **section**.

Their periods are zero.
We only need to compute periods of extensions.

Algorithm for computing periods of elliptic surfaces

1. Compute the set Σ of **critical values**.
2. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \backslash \Sigma, b)$.
3. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
4. Match **boundaries** of thimbles together to obtain **extensions**.
5. Integrate the **periods** on these extensions.

Gauß-Manin connection

The cohomology sheaf $\mathscr{H}^n_{DR}\left(E_t/\mathbb{Q}(t)\right)$ inherits a connection from the derivation in the base $\mathbb{P}^1$: the **Gauß-Manin connection** **[Katz Oda 1968]**.

Period functions $\int_{\gamma_t} \omega_t$ are solutions to a Fuchsian differential equation: **the Picard-Fuchs equation**.

This connection can be computed explicitly via the Griffiths—Dwork reduction **[Griffiths 1969, Dwork 1964]**.

Example: Let $\mathscr{X}_t = V(X^3 + Y^3 + Z^3 + tXYZ)$ be an elliptic surface.
A basis of the De Rham cohomology sheaf is given by the residues of

$$\omega_1(t) = \frac{\Omega}{X^3 + Y^3 + Z^3 + tXYZ} \text{ and } \omega_2(t) = \frac{X^3\Omega}{(X^3 + Y^3 + Z^3 + tXYZ)^2}.$$

Let $\mathscr{L} = (t^3 + 27)\partial_t^2 + 3t^2\partial_t + t$. Then $\mathscr{L}\omega_1$ is an exact differential. In particular for any cycle γ_t the period function $\pi(t) = \int_{\gamma_t} \operatorname{res} \omega_1(t)$ is a solution of $\mathscr{L}$.

Computing monodromy

Globally defined
$\implies$ no monodromy

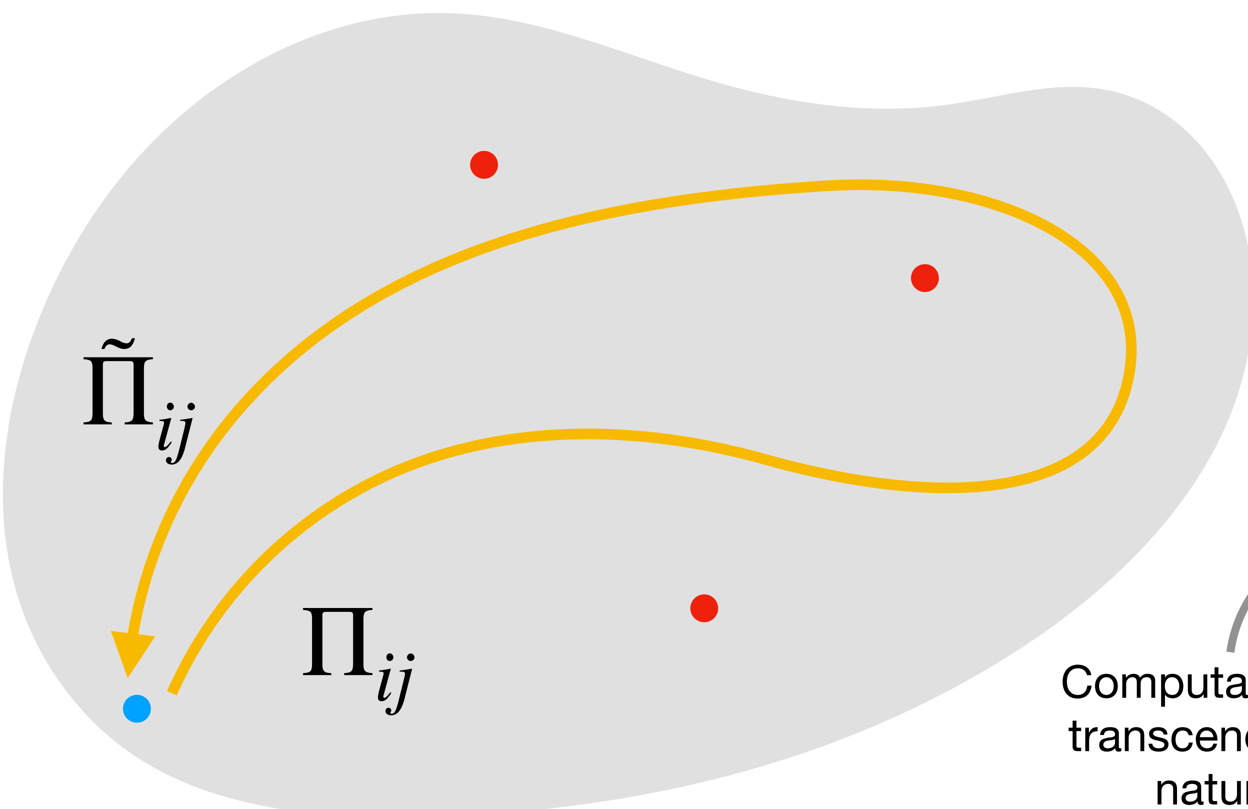
$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

Solution to Picard-Fuchs equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\boxed{\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_2(\mathbb{Z})}$$

Computation of transcendental nature

It is sufficient to carry out this computation with precision $< 1/2$ to recover C exactly.

Computing monodromy of differential operators

[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]

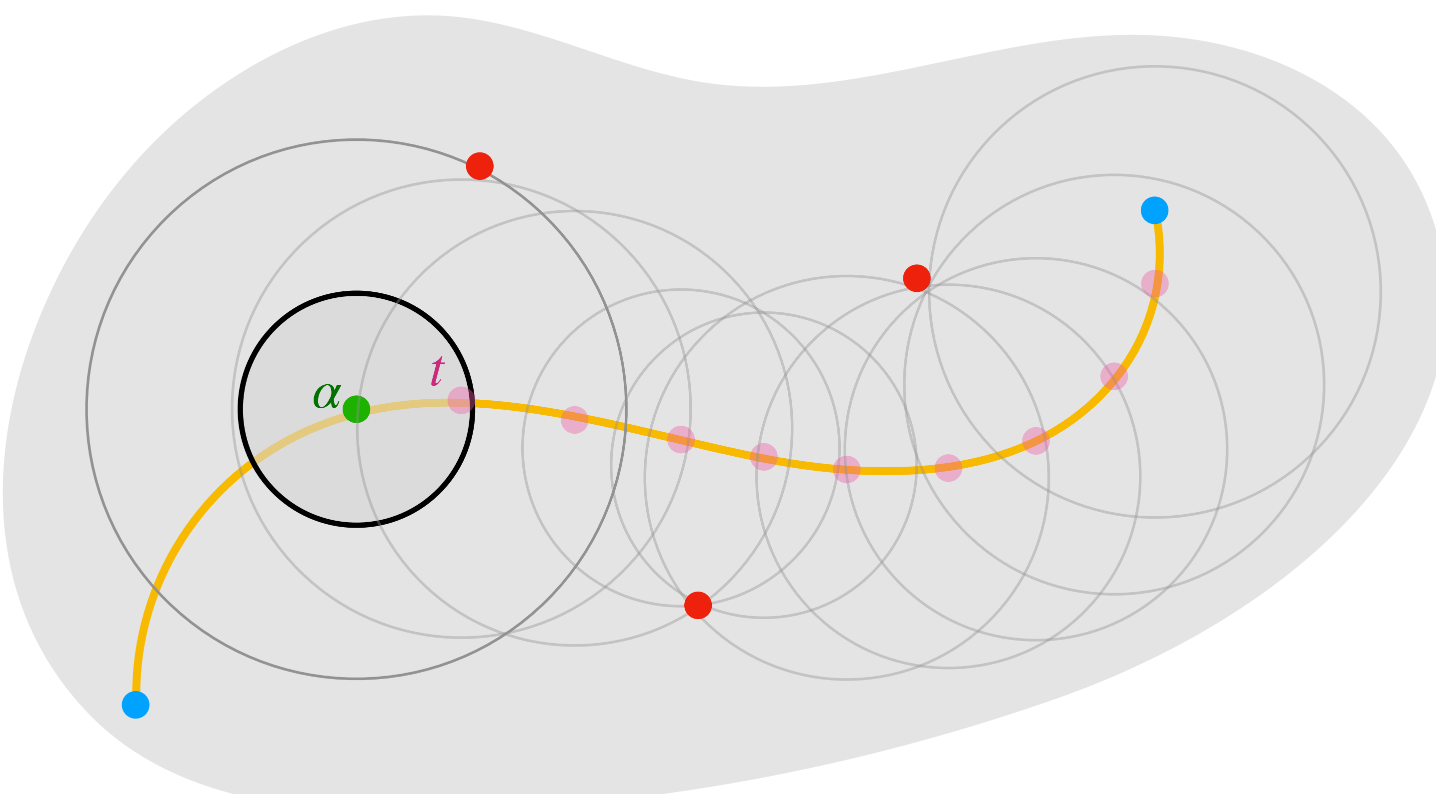
In a small radius around α:

$$\left| f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \right| \leq \mathscr{P}(m)2^{-m}$$

polynomial
in m (effective)
[Mezzarobba Salvy 2009]

We compute $f^{(k)}(\alpha)$ from $\mathscr{L}$.

In a disk around α, the precision given by the Taylor formula is exponential in its order.



From the derivatives at α,
we can recover the derivatives at t.

Linear complexity:
recover m digits in $\mathscr{O}(m)$ operations
(using binary splitting)

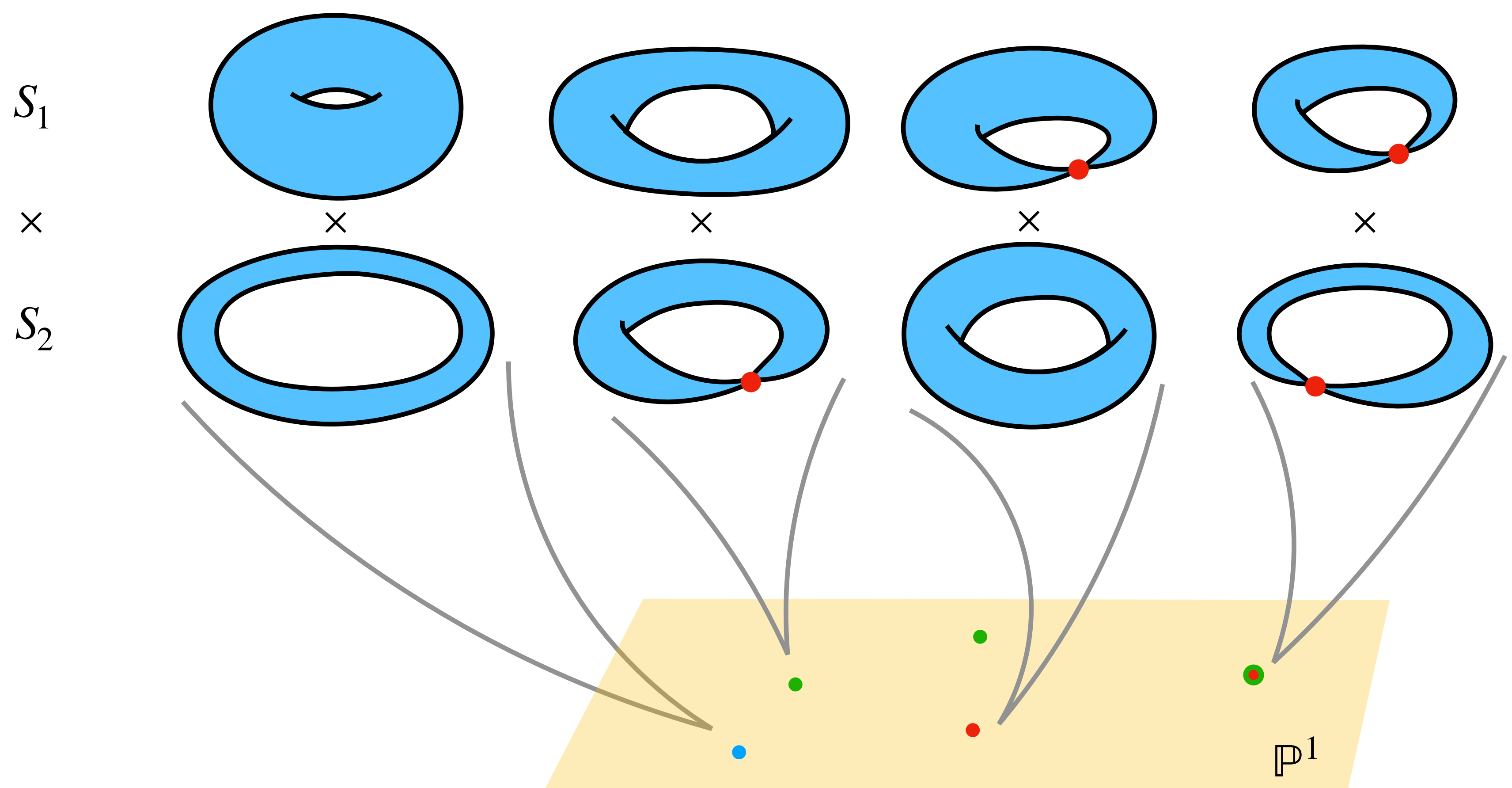
Fibre products

of elliptic surfaces

Schoen's construction

The fibre product $T = S_1 \times_{\mathbb{P}^1} S_2$ of two rational elliptic surface with disjoint critical values yields a smooth **Calabi-Yau threefold. [Schoen 1988]**

"[…] a class of such threefolds which is large enough to exhibit many of the phenomena which one wants to study, yet is special enough to be quite tractable."



When critical values coincide, we obtain a singular threefold.
Under certain conditions, the singularities admit a crepant resolution: we obtain a smooth Calabi-Yau threefold. **[Kapustka² 2009]**

Goal: We want to compute periods of such threefolds

Homology of smooth fibre products

We can use the same construction to compute the parabolic homology.

By the Künneth formula, the homology of the fibre is

$$H_2(E_1 \times E_2) = H_0(E_1) \otimes H_2(E_2) \oplus H_1(E_1) \otimes H_1(E_2) \oplus H_2(E_1) \otimes H_0(E_2)$$

only component with monodromy

Periods of the fibres are products of periods

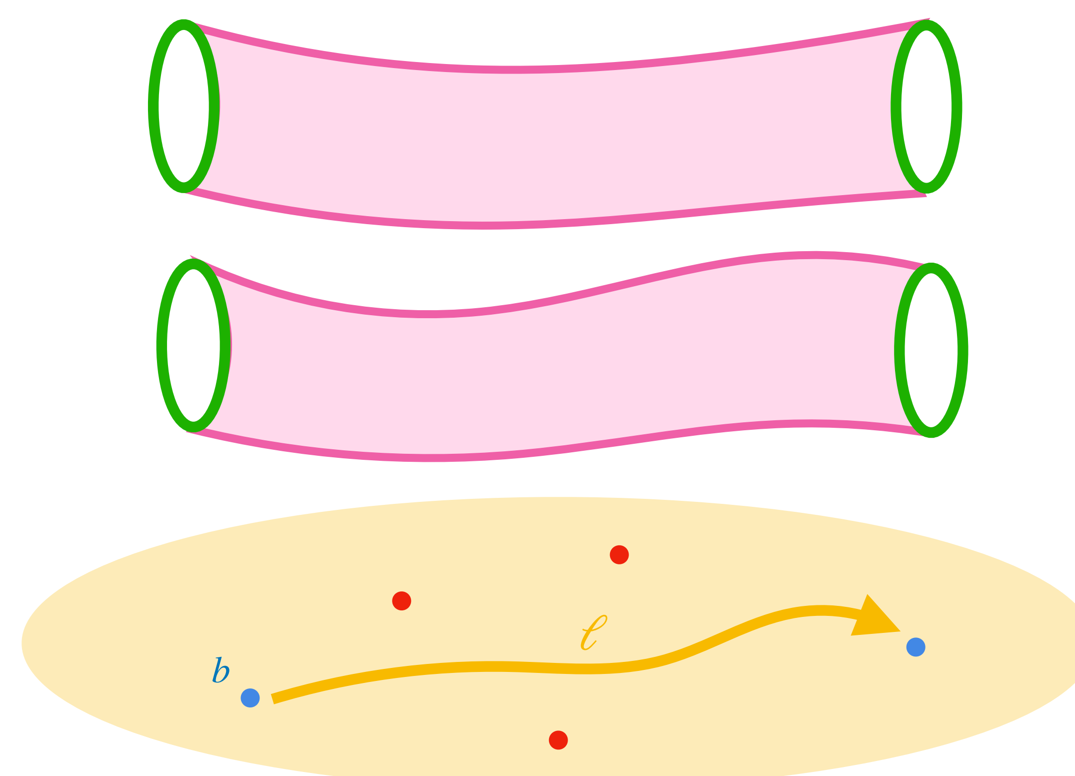
$$\int_{\gamma_1 \times \gamma_2} \omega_1 \otimes \omega_2 = \int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2$$

The monodromy representation on $H_2(E_1 \times E_2)$ is the tensor product of the monodromy representations

$$M_\ell = M_{1\ell} \otimes M_{2\ell} \in \mathrm{GL}_4(Z)$$

$$\int_{\tau_\ell(\gamma_1 \times \gamma_2)} \omega_1 \otimes \omega_2 \wedge \mathrm{d}t = \int_\ell \left(\int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2 \right) \mathrm{d}t$$

Extensions are fibre products of extensions of the elliptic surfaces

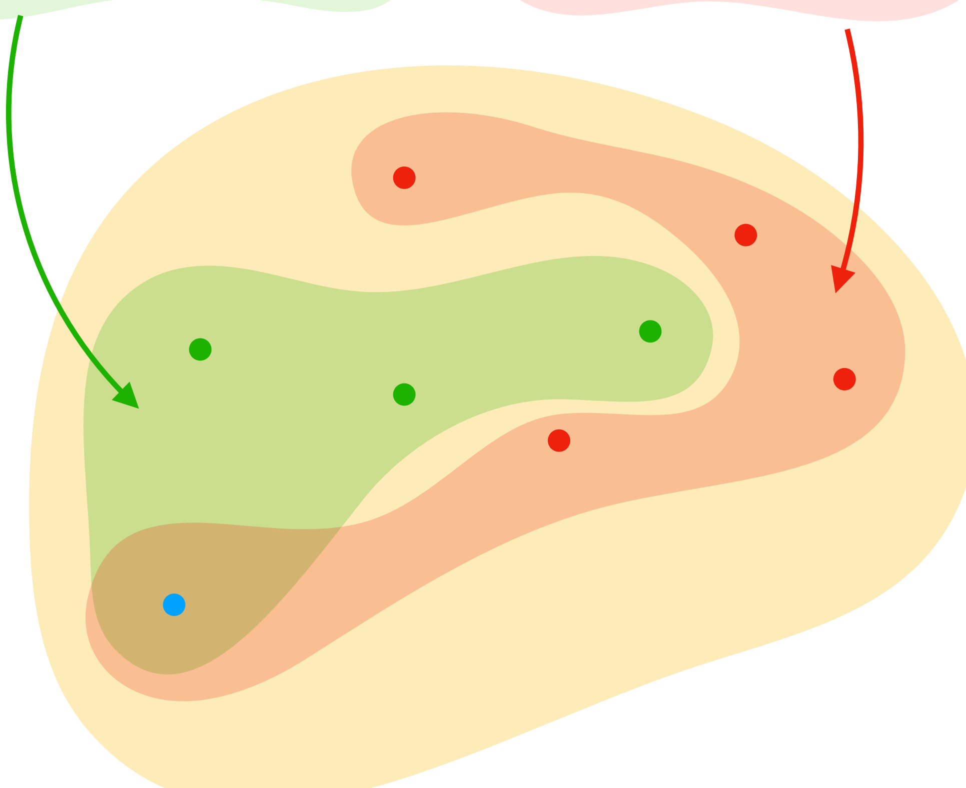


Homology of smooth fibre products

The homology group $H_3(T)$ is generated by **extensions**, and **fibre components** of the elliptic surfaces.

Proposition: the homology of the fibre product can be recovered from the monodromy representation of the elliptic surfaces.

$$H_3(T^*, F_b) = H_2(S_1^*, E_{1b}) \otimes H_1(E_{2b}) \oplus H_2(S_2^*, E_{2b}) \otimes H_1(E_{1b})$$



Here * means that we removed one fibre "at infinity"

In the case of rational surfaces, we recover a result of Schoen:

$H_3(T)$ has rank

$$12 \times 2 + 12 \times 2 - 4 - 4 = 40.$$

$H_2(S_1^*, E_{1b}) \otimes H_1(E_{2b})$

Gluing boundaries

Homologically trivial extensions

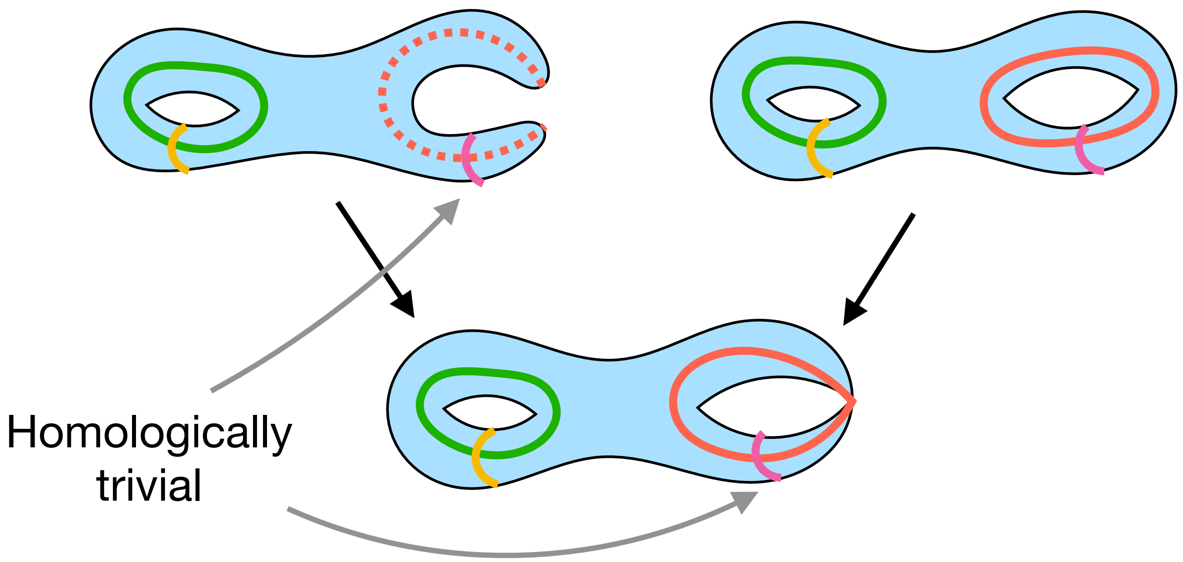
$H_2(S_2^*, E_{2b}) \otimes H_1(E_{1b})$

We have an explicit description of these cycles. We can perform the same integration methods

$$\int_{\tau_\ell(\gamma_1 \times \gamma_2)} \omega_1 \otimes \omega_2 \wedge \mathrm{d}t = \int_\ell \left(\int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2 \right) \mathrm{d}t$$

Smoothings and vanishing cycles

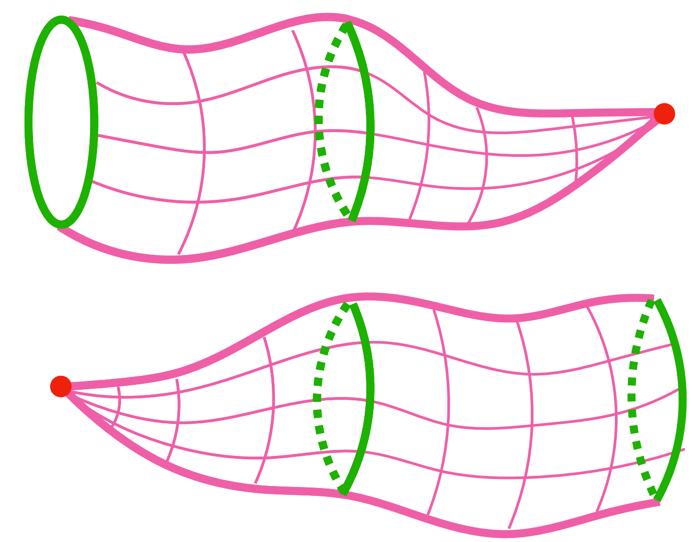
The one dimensional case

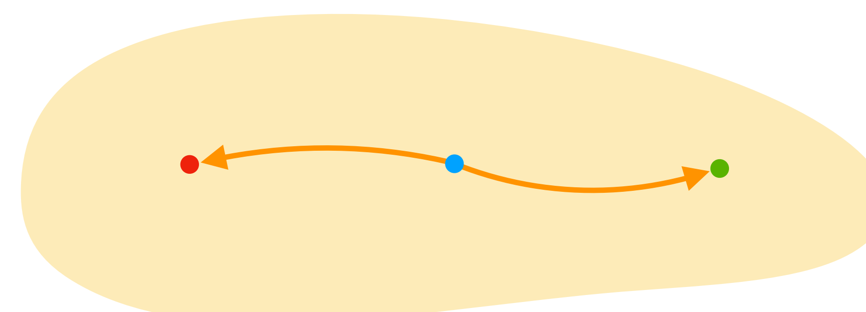


We reduce to the smooth case by **smoothing** the variety.

This creates new cycles which collapse in the singular limit: the **vanishing cycles**.

Our threefolds





We can compute the periods on the orthogonal complement of the lattice of vanishing cycles.

Calabi-Yau operators

Hadamard products

The **Hadamard product** of two elliptic surfaces S_1 and S_2 is the family of threefolds

$$T_u = S_1 \times_u S_2 := S_1 \times_{\mathbb{P}^1} \varphi_u^* S_2$$

where $\varphi_u \colon \mathbb{P}^1 \to \mathbb{P}^1$, $\phi_u(t) = u/t$.

T_u is equipped with a map to $\mathbb{P}^1$ with fibre $E_{1t} \times E_{2\frac{u}{t}}$

Under some condition on the fibres of S_1 and S_2 at 0 and ∞, there is a **maximal unipotent monodromy point** at 0.

The holomorphic period of is the Hadamard product of the periods of the underlying surfaces:

$$\pi(u) = \sum_i a_i b_i u^i \qquad \text{where} \qquad \pi_1(t) = \sum_i a_i t^i \text{ and } \pi_2(t) = \sum_i b_i t^i$$

We are interested in the case where the Picard-Fuchs equation $\mathscr{L}^{\mathrm{Had}}$ has order 4 — see next slides.

T_u is generically singular, but we can smooth homogeneously in u.
It is not known whether we can resolve homogeneously (and crepantly!) in u.

[Golyshev van Straten 2023]

Calabi-Yau operators

Calabi-Yau operators are differential operators in one variable satisfying certain conditions:

→ be **Fuchsian**, i.e. having solutions that have “nice” singularities

→ be **self-dual**, some technical notion stemming from mirror symmetry

→ have a **Maximal unipotent monodromy** point, i.e. such that the monodromy around it satisfies $(M-1)^n = 0$ and $(M-1)^k \neq 0, \forall k < n$

order

→ certain **integrality** conditions on the holomorphic solution, instanton numbers and q-coordinates

These operators are expected to be Picard–Fuchs equations of families of algebraic varieties.

Calabi-Yau operators of order 4

[AESZ 2010] gave a list of around 500 Calabi-Yau (CY) operators of order 4 obtained partially through an extensive computer search.

They are conjectured to be the Picard-Fuchs equations of varieties carrying a motive of type (1,1,1,1).

There are 14 hypergeometric Calabi-Yau operators of order 4 and 105 Hadamard products of elliptic surfaces.

In many cases, a geometric realisation is not known, and in some cases a smooth geometric realisation is not known (e.g. 14th hypergeometric operator).

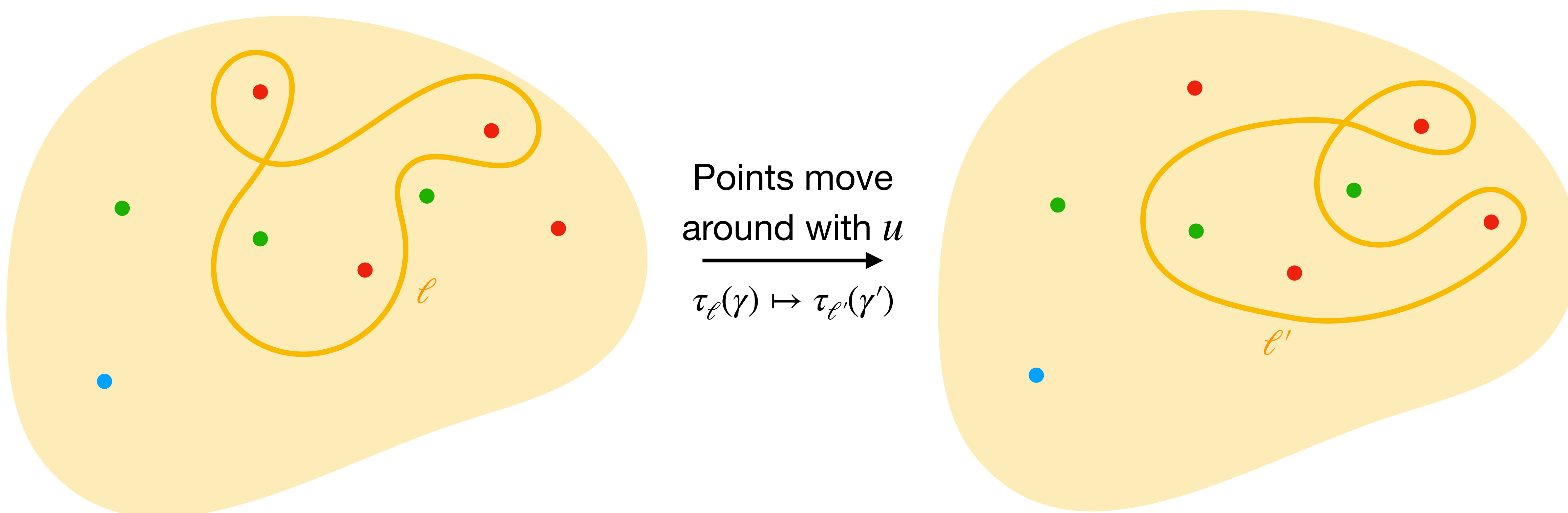
In particular this motivates using smoothings instead of looking for resolutions.

$$
\begin{aligned}
A &: y^2 - yx - ty = x^3 + tx^2\,,\\
B &: y^2 = x^3 - (12t-1)\,x^2 + 48t^2x - 64t^3\,,\\
C &: y^2 = x^3 + (144t-3)\,x - 144t + 2\,,\\
D &: y^2 = x^3 - 3x + 1728t - 2\,,\\
a &: y^2 - (2t-1)\,yx + 3t^2x^2 + 2t^3y + \left(-3t^4\right)x + t^6 = x^3\,,\\
b &: y^2 - (t+11)\,yx - ty = x^3 + tx^2\,,\\
c &: y^2 - (3t-1)\,yx + 3t^2x^2 + 2t^3y = x^3 + 3t^4x - t^6\,,\\
d &: y^2 - (4t-1)\,yx + 2t^2x^2 = x^3 - 4t^4x - (8t^2-8t+1)t^4\,,\\
e &: (16t-1)\,y^2 - (16t-1)\,yx - ty = (16t-1)\,x^3 + tx^2\,,\\
f &: y^2 - (3t-1)\,yx + 9t^3y = x^3 - t^3(6t-1)\left(9t^2-3t+1\right)\,,\\
g &: y^2 - (6t-1)\,yx - 2t^3y = x^3 - 3t^2x^2 + 3t^4x - t^6\,,\\
h &: y^2 = 9x^3 - 3(-1+3t)(-1+27t)^3 - 6(27t-1)^4\left(27t^2+18t-1\right)\,,\\
i &: (64t-1)\,y^2 = (64t-1)\,x^3 - (48t-3)\,x - 16t - 2\,,\\
j &: (432t-1)\,y^2 = (432t-1)\,x^3 - (1296t-3)\,x + 864t + 2\,.
\end{aligned}
$$

Parabolic homology

In all considered cases, $H_3^{\mathrm{para}}(T_u)$ has rank 4 and carries precisely the $(1,1,1,1)$ motive.

The monodromy with respect to u acts by a braid action on $\pi_1(\mathbb{P}^1 \backslash \Sigma_u)$.



The parabolic homology $H_3^{\mathrm{para}}(T_u)$ is stable under monodromy.

In particular the monodromy matrices have integer coefficients.
In other words this realisation of the $(1,1,1,1)$ motive carries a local system defined over $\mathbb{Z}$.

A new Gamma-class formula

[Candelas, De la Ossa, Green, Parkes] An ansatz for the period matrix Π can also be obtained from **topological invariants** of the family from the formula

$$(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi & \frac{c_2H}{24} & 0 & \frac{H^3}{6} \\ \frac{c_2H}{24} & \frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}$$

where $(2i\pi)^i\varpi_i = \sum_{k=0}^{i} h_k(t)\frac{\log^k(t)}{k!}$ form the Frobenius basis at the MUM point

and χ, c_2H, H^3 are the Euler characteristic, the second Chern class and the triple intersection numbers of the mirror threefold.

Using our methods, we can compute this matrix numerically with **very high precision** (several hundred digits) in reasonable time for Hadamard products.

We find a slightly different version which we conjecture to be general:

$$(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2H}{24} - \frac{\delta}{2} & \frac{c_2H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}$$

with intersection product:

$$\begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & N \\ -M & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}$$

where $\chi, c_2H, H^3 \in \mathbb{Z}, \quad \alpha, \delta, \sigma \in \{0,1\}$ and $M, N \in \mathbb{N}$.

It agrees with the above when $M = N = 1$ and $\delta = \alpha = 0$

The 105 Hadamard products

T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$A \times_u A$	-128	184	16	0	0	0	1	1
$A \times_u B$	-144	12	12	0	0	0	1	1
$A \times_u C$	-176	8	8	0	0	0	1	1
$A \times_u D$	-256	4	4	0	0	0	1	1
$A \times_u a$	-120	24	24	0	0	0	2	1
$A \times_u b$	-120	20	20	0	0	0	1	1
$A \times_u c$	-112	0	24	0	0	0	3	1
$A \times_u d$	-88	40	16	0	0	0	4	2
$A \times_u e$	96	112	16	0	1	0	4	1
$A \times_u f$	-120	32	12	0	1	1	3	3
$A \times_u g$	-8	-96	24	0	1	0	6	1
$A \times_u h$	168	216	12	0	1	1	3	1
$A \times_u i$	272	32	8	0	1	0	2	1
$A \times_u j$	472	64	4	0	1	1	1	1
$B \times_u B$	-144	54	9	1	0	0	3	1
$B \times_u C$	-156	48	6	0	0	0	3	1
$B \times_u D$	-204	42	3	1	0	0	3	1
$B \times_u a$	-162	72	18	0	0	0	6	1
$B \times_u b$	-150	66	15	1	0	0	3	1
$B \times_u c$	-156	0	18	0	0	0	3	1
$B \times_u d$	-162	42	12	0	0	0	12	2
$B \times_u e$	24	24	12	0	1	0	12	1
$B \times_u f$	-198	81	9	1	1	1	3	3
$B \times_u g$	-78	54	18	0	1	0	6	1
$B \times_u h$	90	-63	9	1	1	1	3	1
$B \times_u i$	180	-6	6	0	1	0	6	1
$B \times_u j$	342	51	3	1	1	1	3	1
$C \times_u C$	-144	40	4	0	0	0	2	1
$C \times_u D$	-156	32	2	0	0	0	2	1
$C \times_u a$	-228	24	12	0	0	0	2	1
$C \times_u b$	-200	16	10	0	0	0	2	1
$C \times_u c$	-224	72	12	0	0	0	6	1

T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$C \times_u d$	-268	44	8	0	0	0	4	2
$C \times_u e$	-64	32	8	0	1	0	4	1
$C \times_u f$	-312	82	6	0	1	1	6	3
$C \times_u g$	-172	204	12	0	1	0	6	1
$C \times_u h$	0	-126	6	0	1	1	6	1
$C \times_u i$	80	52	4	0	1	0	2	1
$C \times_u j$	208	-10	2	0	1	1	2	1
$D \times_u D$	-120	22	1	1	0	0	1	1
$D \times_u a$	-366	24	6	0	0	0	2	1
$D \times_u b$	-310	14	5	1	0	0	1	1
$D \times_u c$	-364	0	6	0	0	0	3	1
$D \times_u d$	-470	46	4	0	0	0	4	2
$D \times_u e$	-200	40	4	0	1	0	4	1
$D \times_u f$	-534	59	3	1	1	1	3	3
$D \times_u g$	-338	66	6	0	1	0	6	1
$D \times_u h$	-126	27	3	1	1	1	3	1
$D \times_u i$	-44	14	2	0	1	0	2	1
$D \times_u j$	62	25	1	1	1	1	1	1
$a \times_u a$	-72	24	36	0	0	0	2	1
$a \times_u b$	-90	24	30	0	0	0	2	1
$a \times_u c$	-60	72	36	0	0	0	6	1
$a \times_u d$	12	36	24	0	0	0	4	2
$a \times_u e$	216	0	24	0	1	0	4	1
$a \times_u f$	-18	30	18	0	1	1	6	3
$a \times_u g$	96	-396	36	0	1	0	6	1
$a \times_u h$	306	-18	18	0	1	1	6	1
$a \times_u i$	444	12	12	0	1	0	2	1
$a \times_u j$	726	42	6	0	1	1	2	1
$b \times_u b$	-100	22	25	1	0	0	1	1
$b \times_u c$	-80	0	30	0	0	0	3	1
$b \times_u d$	-30	38	20	0	0	0	4	2
$b \times_u e$	160	8	20	0	1	0	4	1
$b \times_u f$	-60	127	15	1	1	1	3	3
$b \times_u g$	50	-390	30	0	1	0	6	1
$b \times_u h$	240	63	15	1	1	1	3	1
$b \times_u i$	360	118	10	0	1	0	2	1
$b \times_u j$	600	-115	5	1	1	1	1	1
$c \times_u c$	-48	0	36	0	0	0	3	1
$c \times_u d$	28	36	24	0	0	0	12	2
$c \times_u e$	224	0	24	0	1	0	12	1

T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$c \times_u f$	0	54	18	0	1	1	3	3
$c \times_u g$	108	36	36	0	1	0	6	1
$c \times_u h$	312	198	18	0	1	1	3	1
$c \times_u i$	448	108	12	0	1	0	6	1
$c \times_u j$	728	-54	6	0	1	1	3	1
$d \times_u d$	80	16	16	0	0	0	4	2
$d \times_u e$	360	40	16	0	0	0	4	2
$d \times_u f$	138	38	12	0	0	1	12	6
$d \times_u g$	236	12	24	0	0	0	12	2
$d \times_u h$	462	54	12	0	0	1	12	2
$d \times_u i$	628	20	8	0	0	0	4	2
$d \times_u j$	986	34	4	0	0	1	4	2
$e \times_u e$	320	64	16	0	0	0	4	1
$e \times_u i$	384	8	8	0	0	0	4	1
$e \times_u j$	528	28	4	0	0	1	4	1
$f \times_u e$	384	20	12	0	0	1	12	3
$f \times_u f$	36	6	9	1	0	0	3	3
$f \times_u g$	234	0	18	0	0	1	6	3
$f \times_u h$	504	6	9	1	0	0	3	3
$f \times_u i$	696	40	6	0	0	1	6	3
$f \times_u j$	1104	2	3	1	0	0	3	3
$g \times_u e$	328	120	24	0	0	0	12	1
$g \times_u g$	264	72	36	0	0	0	6	1
$g \times_u h$	390	0	18	0	0	1	6	1
$g \times_u i$	500	24	12	0	0	0	6	1
$g \times_u j$	754	48	6	0	0	1	6	1
$h \times_u e$	336	180	12	0	0	1	12	1
$h \times_u h$	324	54	9	1	0	0	3	1
$h \times_u i$	336	72	6	0	0	1	6	1
$h \times_u j$	420	18	3	1	0	0	3	1
$i \times_u i$	304	40	4	0	0	0	2	1
$i \times_u j$	320	8	2	0	0	1	2	1
$j \times_u j$	244	22	1	1	0	0	1	1

The AESZ list

The Gamma-class formula

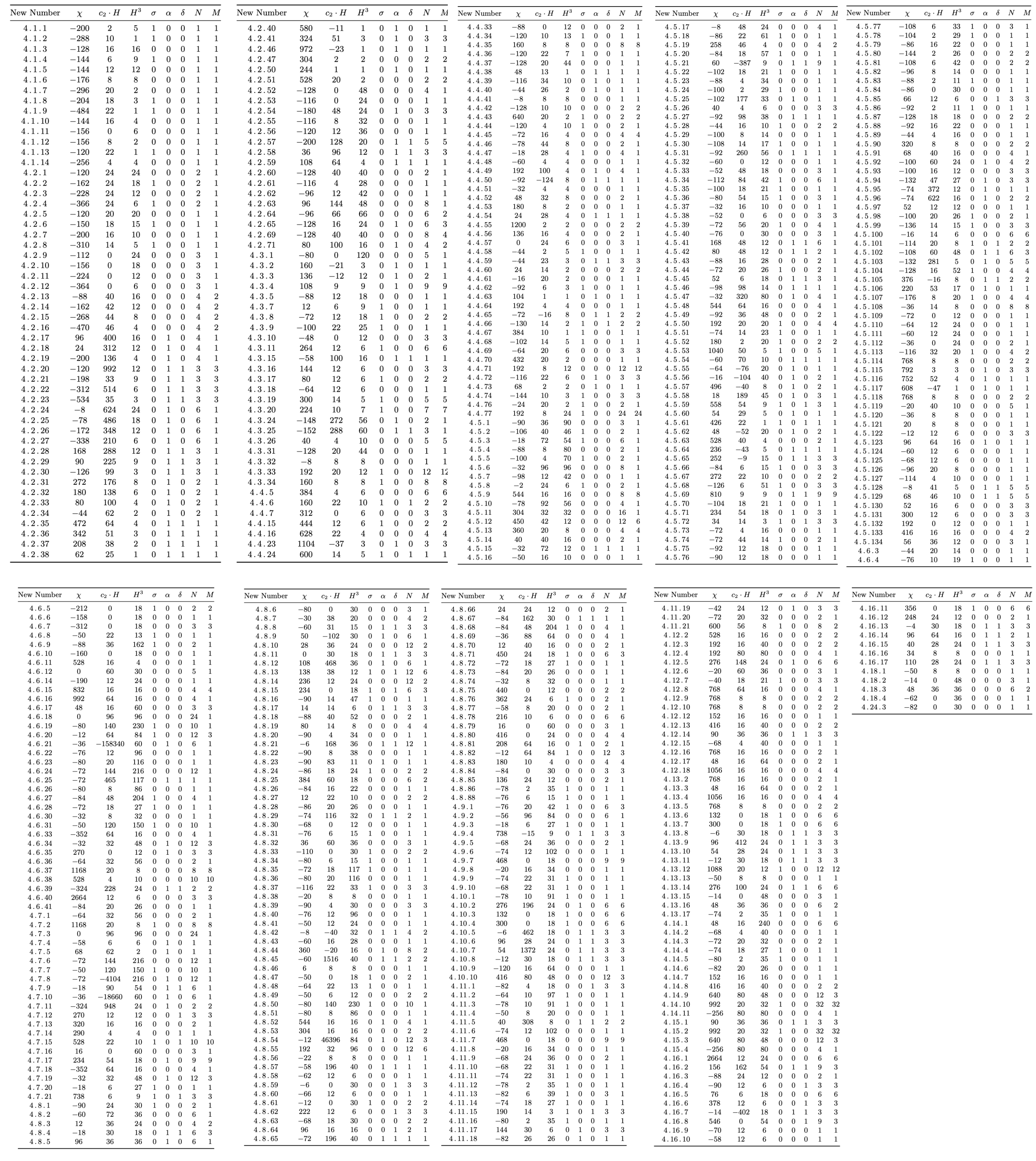
$$
(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2H}{24} - \frac{\delta}{2} & \frac{c_2H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}
$$

seems to apply to **all** operators in the CYDB (at least up to order 20).

“[…] a class of such threefolds which is large enough to exhibit many of the phenomena which one wants to study, yet is special enough to be quite tractable.”

In some cases $M \neq N$ – in particular the operator seems to not have $\mathrm{Sp}_4(\mathbb{Z})$-integral monodromy.

$$
\begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & N \\ -M & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}
$$



See also **[Yang 2020]**

The Deligne conjecture

with Nutsa Gegelia
and Duco van Straten

Another application concerns **numerical checks of the Deligne conjecture (1979)**, which relates a minor c^+ of the period matrix to the value $L(2)$ of the L-function via the formula $L(2) = qc^+$, where $q \in \mathbb{Q}$.

Fibre product	ratio
$A \times_1 A$	-2^{-4}
$A \times_1 B$	$2^2 \cdot 3^{-2}$
$A \times_1 c$	3^{-1}
$A \times_1 d$	2^{-2}
$B \times_1 B$	$2^8 \cdot 3^{-5}$
$B \times_1 c$	$-2^5 \cdot 3^{-3}$
$A \times_{-1} A$	-2^{-4}

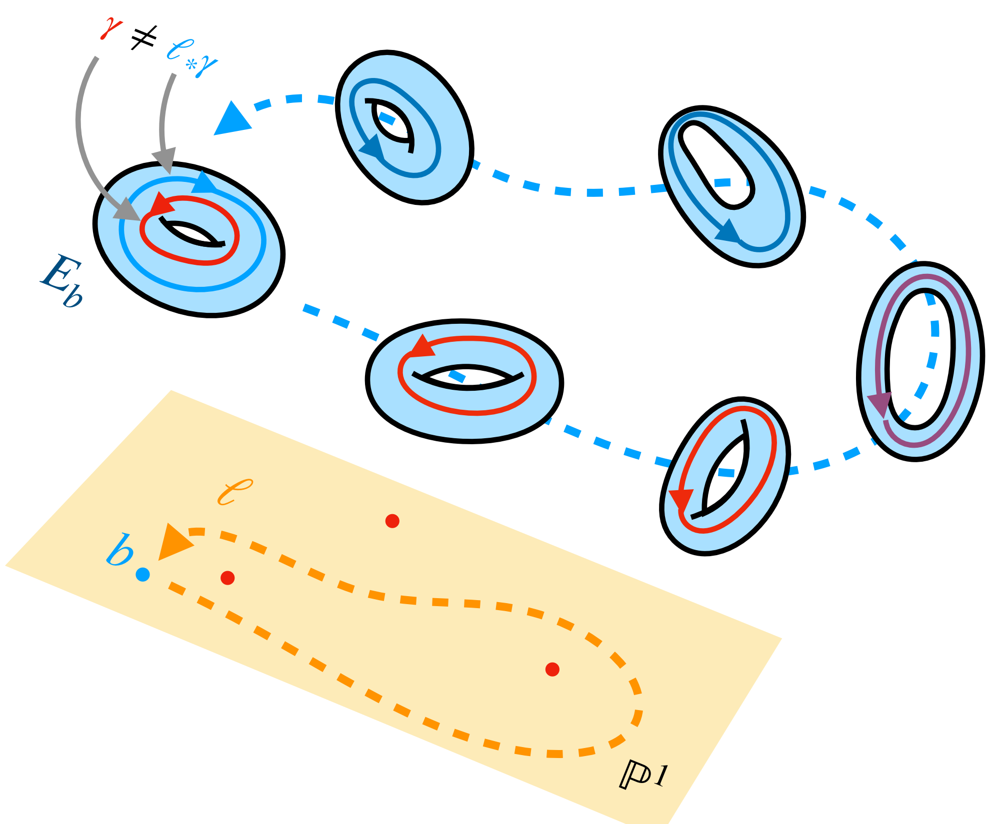
Fibre product	ratio
$A \times_{-1} B$	$2^2 \cdot 3^{-2}$
$A \times_{-1} b$	2^{-5}
$A \times_{-1} c$	3^{-1}
$A \times_{-1} f$	$-2 \cdot 3^{-1}$
$B \times_{-1} B$	$2^8 \cdot 3^{-4}$
$B \times_{-1} c$	$2^6 \cdot 3^{-3}$
$B \times_{-1/8} a$	$7 \cdot 3^2 \cdot 2^{-2}$

We are able to numerically recover the value of q for several examples with many digits of precision.

Why?

In many examples of Hadamard products of elliptic surfaces the L-value vanishes. Instead the Beilinson conjecture applies.

Thank you!



$$\begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2 H}{24} - \frac{\delta}{2} & \frac{c_2 H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2 H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix}$$