Towards HMS for log dP surfaces

(w/ G. Gugiatti)

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§ Motivation: Fano/LG mirror symmetry

General idea: $X \quad \rightsquigarrow \quad Y \xrightarrow{w} \mathbb{C}$

Fano orbifold   fibration

A sympl.   A   A

B complex   B   B

Expressed in different ways, e.g.:

(i) cohomological data

(e.g. reg'd q. period $\hat{G}_X(t) \rightsquigarrow \pi(t)$ classical period
(q. cohom, GW invts' A)   on Y (measure VHS, B)

(ii) Categorical data (Homological Mirror Symmetry, HMS):

$$\underset{B}{D^b(\mathrm{Coh}\,X)} \simeq \underset{A}{FS(Y,w)}$$

[Kontsevic]

Long term goal: prove (ii) when X is a Johnson-Kollár surface:

- these are log del Pezzo: $-K_X$ ample, quotient sing's
- assume anticanonical & quasi-smooth

[JK] $\Longrightarrow$ all belong to a finite list (23) OR to a countable series

JK Series: $k \in \mathbb{Z}_{>0}$.

$$X \in |\mathcal{O}_{\mathbb{P}}(8k+4)|, \quad \mathbb{P} = \mathbb{P}(\overset{x}{2}, \overset{y_1}{2k+1}, \overset{y_2}{2k+1}, \overset{z}{4k+1})$$

$$X = \{ a x^{4k+2} + h_4 + b x z^2 + x^{2k+1} h_2 + x^{k+1} h_1 z = 0 \}$$

(with $h_i = h_i(y_1, y_2)$ deg i)

Properties: • X Fano: $-K_X = \mathcal{O}(1)$, & $|-K_X| = \emptyset$ (!)

$\Downarrow$

• no standard mirror $\leftsquigarrow$ no toric degeneration (Corti Coates Galkin, Gross Siebert)

• Corti Gugiatti '21: construct cohomological (hypergeometric) mirrors.

Goal: Prove that CG'21 mirrors are homological

Today: ① $D^b \operatorname{Coh} X$ ← orbifold

② Some JK cases. $k=0$ & dP_1, dP_2, dP_3.

↑ we'll look at some mirrors (i) & (ii) & compare them

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§ $D^b \operatorname{Coh} X$ [Thm [Gugiatti - R.] $D^b \operatorname{Coh} X$ admits an explicit full exc. collection.]

Morally: $D^b \operatorname{Coh} X$ can be broken down in simple pieces, as explicitly as we want.

Recall: $\mathcal{D}$ Δ'd category.

- $E \in \mathcal{D}$ is exceptional if $\mathrm{Ext}^*(E,E) = \mathbb{C}[0]$
- $(E_1, \dots, E_m)$ exc. collection if
  - E_i exc. $\forall i$
  - $\mathrm{Ext}^*(E_i, E_j) = 0 \quad \forall j < i$
- **full** if $\mathcal{D}$ is the smallest Δ'd subcat. containing all the E_i.

Proof Outline: [Ishii-Ueda] GL_2-McKay correspondence:

min'l resolution of X

$\exists$ fully faithful $\phi : D^b(\tilde{X}) \longrightarrow D^b(\mathcal{X})$ & a (semiorthogonal) decomposition

$$\boxed{\begin{array}{c} D^b(\mathcal{X}) = \langle \underbrace{e_1, \dots, e_N}_{b}, \underbrace{\phi D^b(\tilde{X})}_{a} \rangle \\ c : \mathrm{Hom}(e_i, \phi D^b(\tilde{X})) \end{array}}$$

b is combinatorial [Ishii]

The min'l resolution $\tilde{X}$:

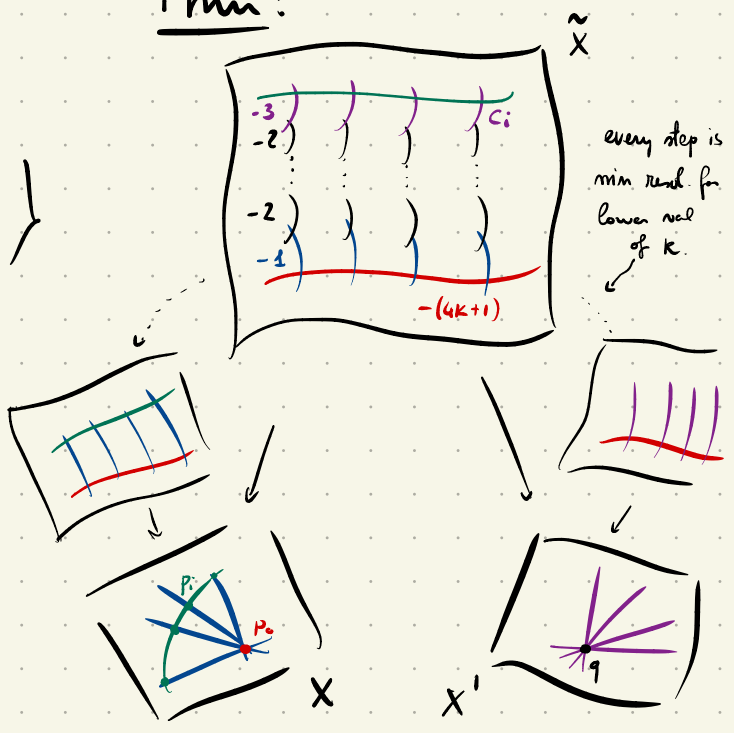
Singularities of X:

- $p_0 \circlearrowright \frac{1}{4k+1}(1,1) = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \middle| \zeta^{4k+1} = 1 \right\}$
- $p_1\ p_2\ p_3\ p_4 \circlearrowright \frac{1}{2k+1}(1,k)$

Fact: X' is a dP_2 with a special point q (codim 2)

Eckardt

Thm:



Cor: The moduli space of surface X_k is isomorphic to the locus of dP2 surfaces w/ Eckardt pt.

(Note: we could guess that all were isomorphic)

Q: How does this reflect on the mirror?

Ⓒ $\mathrm{Hom}_{\mathcal{X}}(e_i, \phi F) = \mathrm{Hom}_{\tilde{X}}(\psi e_i, F)$

- easier (no stack)
- local on exc. locus of $\tilde{X}$
  (intrinsic to the singularity)

Rmk: Last wrinkle: to write $D\tilde{X}$ explicitly, we need to classify sets of 7 skew lines in X'
(equiv., sets of 7 pts in $\mathbb{P}^2$ giving Eck. pts)

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§ Mirrors (low k: in progress) (our recent/upcoming work)

e.g.: $M = \{ y^2 = (4a + x^{2k+1})(a^2 x - 4(4a + x^{2k+1})^2) \} \subseteq (\mathbb{C}^*)^3_{x,y,a}$

$$\downarrow$$

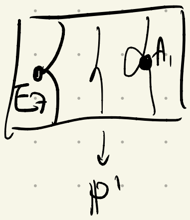
$$\mathbb{C}^*_a$$

i.e. $y^2 =$ cubic(a) (fixed x)
$(6k+3)$-ic(x) (for fixed a)

$k \geq 1$ in progress

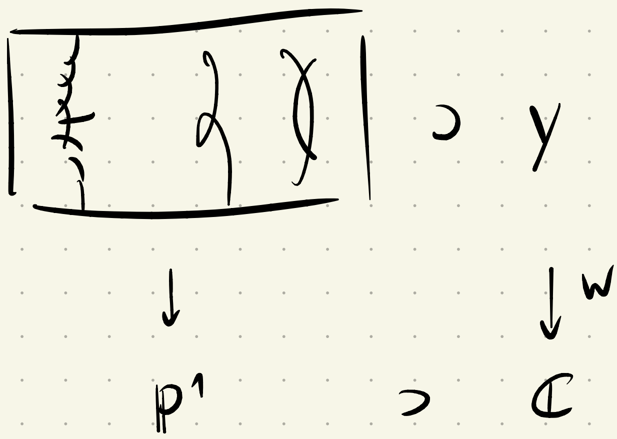
$k = 0$ the story collapses: X is a dP2 & $M \to \mathbb{C}^*$ a LG model matching the HV mirror.

From $M \to \mathbb{C}^* \leftarrow$ Weierstrass equation

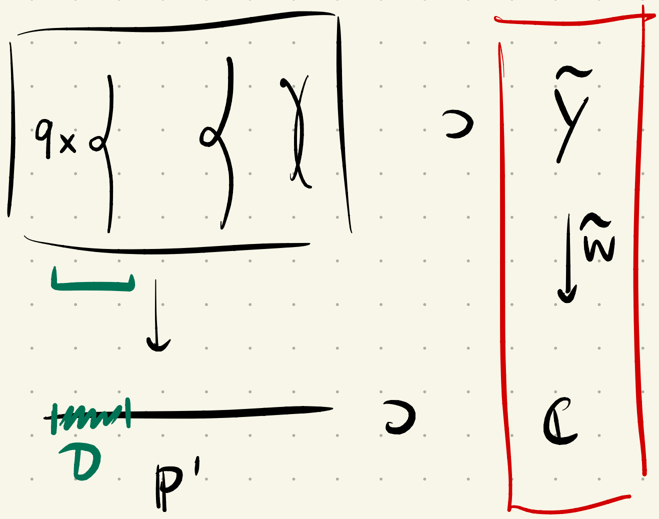


we can:

- obtain an elliptic surface (compactify)

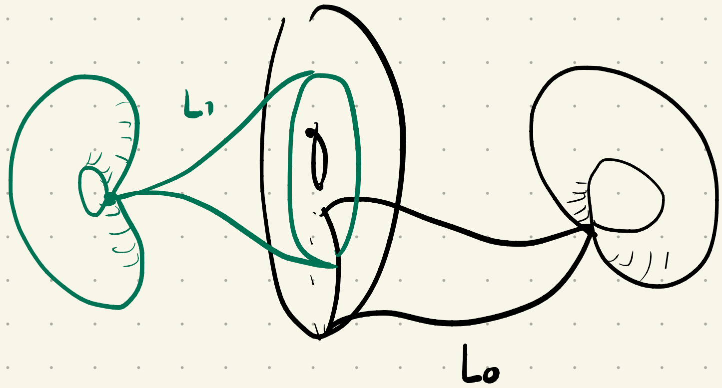


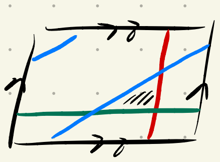
- deform & compactify:

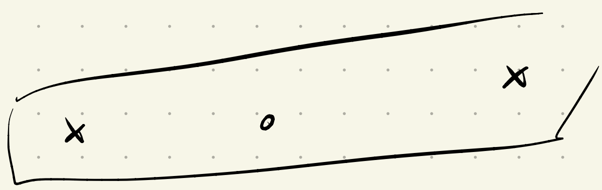


From the Lefschetz fibrations we extract Fukaya-Seidel categories $FS(\tilde{Y}, \tilde{w})$:

- thimbles are exc. objects
- Hom's are $L_i \cap L_j$
- composition law are symplectic areas:







Thm ① We compute an exc. coll. of $FS(\tilde{Y}, \tilde{w})$, $(L_0, \dots, L_9)$

② we find a seq. of mutations which realize

$$\begin{array}{ccc} K_0(FS(\tilde{Y}, \tilde{w})) & \simeq & K_0(Coh(X)) \\ \cup & & \cup \text{ Orlov} \\ K_0(L_0^{\perp}) & \simeq & K_0(MF(f)) \end{array}$$

← Predicted by [Harder - Thompson]

($X = \{f = 0\}$

③ The above also for dP_1, dP_3 :

$$\begin{array}{ccc} dP_1 & \text{blowdowns} & \\ \downarrow & & \\ dP_2 & \to & dP_3 \end{array} \quad \xleftrightarrow{\text{M.S.}} \quad Y_1 \overset{\text{specializations}}{\rightsquigarrow} Y_2 \rightsquigarrow Y_3$$