# REFINEMENT OF THE INFINITESIMAL VARIATION OF HODGE STRUCTURE: THE CASE OF CANONICAL CURVES
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**Abstract**

Let $C$ be a smooth complex projective curve with canonical divisor $K_C$ very ample. The general objective of the research is to study the relation between the cup-product

$$H^1(\Theta_C) \longrightarrow H^0(\mathcal{O}_C(K_C))^* \otimes H^1(\mathcal{O}_C)$$

where $\Theta_C = \mathcal{O}_C(-K_C)$ is the holomorphic tangent bundle of $C$, and the geometry of the canonical embedding of $C$. The cup-product, following Griffiths, stratifies $\mathbb{P}(H^1(\Theta_C))$ by the subvarieties $\Sigma_r$ according to the rank $r$ of $\xi \in H^1(\Theta_C)$, viewed as the linear map

$$\xi : H^0(\mathcal{O}_C(K_C)) \longrightarrow H^1(\mathcal{O}_C),$$

or, equivalently, by the dimension of the kernel of $\xi$

$$W_\xi = ker(\xi).$$

The pair $(\xi, W_\xi)$ is one of the invariants of the Infinitesimal Variation of Hodge Structure (IVHS) of Griffiths and his collaborators. We construct a refinement of $(\xi, W_\xi)$ by attaching an intrinsic filtration $W_\xi^\bullet([\phi])$ of $W_\xi$, varying holomorphically with $[\phi] \in \mathbb{P}(W_\xi)$. The filtration has geometric meaning:

1) it is related to special divisors on $C$,

2) it 'counts' certain rational normal curves in the canonical embedding of $C$.

The filtration arises from the skew-symmetric pairing

$$\alpha_\xi^{(2)} : \bigwedge^2 W_\xi \longrightarrow H^0(\mathcal{O}_C(K_C))$$

naturally attached to the IVHS pair $(\xi, W_\xi)$; this could be considered as a 'symplectic' structure on $W_\xi$.

It is shown that our refinement gives the well known results about the strata $\Sigma_0$ and $\Sigma_1$:

- $\Sigma_0$ is empty, if $C$ has no $g_2^1$,

- the stratum $\Sigma_1$ is the image of the bicanonical embedding of $C$ as long as $C$ has no $g_2^1$, $g_3^1$, $g_5^2$.

That is, one recovers as corollaries the classical theorems of Max Noether on projective normality of the canonical embedding and Babbage-Enriques-Petri about the canonical curve being cut out by quadrics.

Conceptually, the refinement concerns the incidence correspondence

$$P = \{([\xi],[\phi]) \in \mathbb{P}(H^1(\Theta_C)) \times \mathbb{P}(H^0(\mathcal{O}_C(K_C)))|\xi\phi = 0 \text{ in } H^1(\mathcal{O}_C)\}$$

which comes with the projections on each factor

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}(H^1(\Theta_C)) & & \mathbb{P}(H^0(\mathcal{O}_C(K_C))) \end{tikzcd}$$

The left side of the diagram controls the (infinitesimal) variations of complex structure on $C$. The refinement exhibits additional structures on the fibres of $p_1$. In particular, one obtains the stratification of the inverse images $p^{-1}(\Sigma_r)$ of the Griffiths' strata $\Sigma_r$ into substrata where the length of the filtrations $W^{\bullet}_{\xi}(\phi)$, $([\xi],[\phi]) \in p^{-1}(\Sigma_r)$, remains constant. On each such substratum new aspects emerge:

- quiver representations,
- Fano toric variety with a distinguished anti-canonical divisor,
- dimer models.

The quiver emerges from the construction and properties of the refinement; the Fano toric variety arises formally from the graph underling the quiver, but it also has a meaning of moduli of a sort of Higgs structures of the linear algebra of the refinement. The graph underlying the refinement becomes an important part of the theory: it connects to topics such as the Topological Quantum field theory, moduli spaces of elliptic curves with marked points, modular curves, higher categorical structures.

# § 0 Introduction

This a an expanded version of the lecture I gave at the conference. It is based on the ongoing research. Many pieces are far from being understood. But the picture which emerges seems to be interesting enough and related to the topics of the conference: Calabi-Yau varieties, elliptic curves, connections to physics/mirror symmetry. Let me start by recalling some basics about the Infinitesimal Variation of Hodge structure.

About forty years ago, in a series of papers Griffiths and his collaborators proposed several constructions naturally attached to the Infinitesimal Variations of Hodge structure, the IVHS constructions or invariants for short, see [CGrGH], [GH], [G] and also [G1], for an overview and additional references on the subject. More precisely, let $X$ be a complex projective manifold of complex dimension $n$; denote by $\Theta_X$ (resp. $\Omega_X$) the holomorphic tangent (resp. cotangent) bundle of $X$. Then the period map of the variation of Hodge structure on the cohomology space $H^n(X,\mathbf{C})$ has the differential which can be defined as the cohomology cup product

$$H^1(\Theta_X) \longrightarrow \bigoplus_{p+q=n} Hom(H^q(\Omega^p_X), H^{q+1}(\Omega^{p-1}_X))$$

coming from the obvious contraction morphism

$$\Theta_X \longrightarrow \mathcal{H}om(\Omega^p_X, \Omega^{p-1}_X),$$

where $\Omega^a_X = \bigwedge^a \Omega_X$ is the sheaf of germs of holomorphic $a$-forms on $X$. As an illustration and one of the successful applications of the IVHS constructions one often cites the case $n = 1$, that is $X$ is a smooth complex projective curve and we switch to using $C$ instead of $X$. The above cup product becomes

$$H^1(\Theta_C) \longrightarrow Hom(H^0(\Omega_C), H^1(\mathcal{O}_C)).$$

Here the cotangent bundle $\Omega_C$ is the canonical line bundle of $C$ which we write $\mathcal{O}_C(K_C)$, where $K_C$ is the canonical divisor class of $C$, the tangent bundle $\Theta_C = \mathcal{O}_X(-K_C)$, the dual of $\mathcal{O}_C(K_C)$, and the above cup product becomes

$$\begin{aligned} H^1(\mathcal{O}_C(-K_C)) \longrightarrow Hom(H^0(\mathcal{O}_C(K_C)), H^1(\mathcal{O}_C)) &= H^0(\mathcal{O}_C(K_C))^* \otimes H^1(\mathcal{O}_C) \\ &\cong H^0(\mathcal{O}_C(K_C))^* \otimes H^0(\mathcal{O}_C(K_C))^*, \end{aligned} \tag{0.1}$$

where the last equality comes from Serre duality $H^1(\mathcal{O}_C) \cong H^0(\mathcal{O}_C(K_C))^*$.

One of the insights of Griffiths about the differential of the period map could be summarized as follows:

the study of the *degeneracies* of the cup product involved in the differential of the period map leads to important facts about geometry of the projective varieties as well as their moduli spaces.

In fact, Griffiths in [G] attaches to the cup product (0.1) the determinantal varieties

$$\Sigma_r := \{[\xi] \in \mathbb{P}(H^1(\Theta_C)) |\ \xi : H^0(\mathcal{O}_C(K_C)) \longrightarrow H^1(\mathcal{O}_C),\ rk(\xi) \leq r\}, \tag{0.2}$$

for all $r \in [0, g]$, which are the degeneracy loci of the morphism of sheaves

$$H^0(\mathcal{O}_C(K_C)) \otimes \mathcal{O}_{\mathbb{P}(H^1(\Theta_C))}(-1) \longrightarrow H^1(\mathcal{O}_C) \otimes \mathcal{O}_{\mathbb{P}(H^1(\Theta_C))}.$$

This gives the stratification

$$\mathbb{P}(H^1(\Theta_C)) = \Sigma_g \supset \Sigma_{g-1} \supset \cdots \supset \Sigma_1 \supset \Sigma_0. \tag{0.3}$$

The stratum $\Sigma_0$ controls the kernel of the differential of the period map. Thus the statement

$$\Sigma_0 \textit{ is empty} \tag{0.4}$$

is equivalent to the Infinitesimal Torelli for curves. The dual version of (0.1) together with (0.4) becomes the classical theorem of Max Noether: the canonical embedding of a curve is projectively normal, see [G-H]. Thus the knowledge about the degeneracy stratum $\Sigma_0$ is equivalent to an important geometric fact - the projective normality of the canonical embedding, *and* the Infinitesimal Torelli theorem - the period map is an immersion.

Griffiths goes further and shows that the next stratum, $\Sigma_1$, is precisely the image of $C$ under the bicanonical morphism, the one defined by $\mathcal{O}_C(2K_C)$, provided one knows another fundamental fact in the theory of curves, the Babbage-Enriques-Petri theorem. That theorem says that for smooth, complex projective curves with no special linear systems $g^1_2$, $g^1_3$, $g^2_5$, the canonical curve is determined as the intersection of all quadrics passing through it, see [StD], also [M] for an inspiring overview. Thus Griffiths theorem

$$\{\ C \textit{ has no } g^1_2,\ g^1_3,\ g^2_5\} \Leftrightarrow \{\Sigma_1 = \textit{the bicanonical image of } C\}$$

tells us that the degeneracy locus $\Sigma_1$ is tied to the important fact about the geometry of the canonical embedding, the ideal is generated by quadrics, and provides a proof of the generic global Torelli theorem for curves.

One would expect that the higher strata in (0.3), in a similar way, contain interesting facts about the geometry of curves as well as their moduli spaces. It is with this thinking that the author embarked on the investigations of the strata $\Sigma_r$. In this work we propose a refinement of the Griffiths stratification (0.3). This, in particular, will recover known results about the strata $\Sigma_0$ and $\Sigma_1$, *independently* of the old classical theorems quoted above.

## § 1 Refinement of IVHS - the main result

The refinement concerns the Kodaira-Spencer classes of rank $r$ parametrized by the locally closed strata

$$\Sigma_r^0 := \Sigma_r \setminus \Sigma_{r-1}.$$

By definition each stratum $\Sigma_r^0$ comes equipped with a distinguished vector bundle of rank $(g-r)$ whose fibre at a point $[\xi] \in \Sigma_r^0$ is the subspace

$$W_\xi := ker\Big(H^0(\mathcal{O}_C(K_C)) \xrightarrow{\xi} H^1(\mathcal{O}_C)\Big),$$

of the canonical space $H^0(\mathcal{O}_C(K_C))$. For every nonzero $\phi \in W_\xi$ our refinement attaches an intrinsic filtration of $W_\xi$

$$W_\xi = W_\xi^0([\phi]) \supset W_\xi^1([\phi]) \supset \cdots \supset W_\xi^l([\phi]) \supset W_\xi^{l+1}([\phi]) = 0. \tag{1.1}$$

Some of the properties of this filtration are summarized in the following statement.

**Theorem 1.1** *The filtration* (1.1) *is subject to the following properties.*

• *$W_\xi^l([\phi]) \supset \mathbf{C}\phi$ and if the inclusion is strict, then there is a special line bundle $\mathcal{L}$ on $C$ such that $H^0(\mathcal{L})$ is at least two dimensional: there is a plane $P_\phi$ in $W_\xi^l([\phi])$ containing $\mathbf{C}\phi$ and injecting into $H^0(\mathcal{L})$.*

• *For every $i \in [2, l-1]$ (one assumes here that $l \geq 3$), the graded pieces $W_\xi^i([\phi])/W_\xi^{i+1}([\phi])$ of the filtration parametrize cones over rational normal curves of degree $i$ containing the hyperplane section*

$$Z_\phi = (\phi = 0) \subset C \subset \mathbb{P}(H^0(\mathcal{O}_C(K_C))^*)$$

*of $C$ in its canonical embedding.*

This could be viewed as a generalization of the features encountered in the discussion of the strata $\Sigma_i$, for $i = 0, 1$. Namely, the line bundle in the first item of the theorem should be viewed as a special linear system on $C$ (in the case of $\Sigma_0$ it is $g_2^1$, and for $\Sigma_1$ these are $g_2^1$, $g_3^1$, $g_5^2$ of the classical theorem of Babbage-Enriques-Petri), and the second item is related to quadrics through the canonical image of $C$.

More conceptually, our refinement takes place on the cohomological incidence correspondence

$$P = \{([\xi], [\phi]) \in \mathbb{P}(H^1(\Theta_C)) \times \mathbb{P}(H^0(\mathcal{O}_C(K_C))) | \xi\phi = 0 \text{ in } H^1(\mathcal{O}_C)\}$$

which comes with the projections on each factor

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}(H^1(\Theta_C)) & & \mathbb{P}(H^0(\mathcal{O}_C(K_C))) \end{tikzcd} \tag{1.2}$$

The left side of the diagram controls the (infinitesimal) variations of complex structure on $C$ with the fibre over a point $[\xi]$ being the projectivization of the space $W_{\xi}$. The right hand side of the diagram is related to the canonical embedding of $C$ via the geometric incidence correspondence

$$\mathcal{Z}_C = \{([\phi], x) \in \mathbb{P}(H^0(\mathcal{O}_C(K_C))) \times C | \phi(x) = 0\}$$

and its two projections

$$\begin{tikzcd} & \mathcal{Z}_C \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathbb{P}(H^0(\mathcal{O}_C(K_C))) & & C \subset \mathbb{P}(H^0(\mathcal{O}_C(K_C))^*) \end{tikzcd}$$

Putting the two incidences together gives the diagram

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \mathcal{Z}_C \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ \mathbb{P}(H^1(\Theta_C)) & & \mathbb{P}(H^0(\mathcal{O}_C(K_C))) & & C \subset \mathbb{P}(H^0(\mathcal{O}_C(K_C))^*) \end{tikzcd} \tag{1.3}$$

On the left side we have the cohomological incidence correspondence and on the right side the geometric correspondence. One of the goals of IVHS constructions could be summarized as gaining information on the geometric side from relevant data on the cohomological side.

Theorem 1.1 provides an extra structure along the fibres of the projection $p_1$ on the cohomological side and gives a partial translation of these data in terms of the geometry of $C$ and its canonical embedding. One can make a case, and this is done in [R5], that our refinement is a sort of algebraic 'Kähler structure' on $W_{\xi}$ depending on $[\phi]$ in $\mathbb{P}(W_{\xi})$, a sort of hard Lefschetz decomposition of $W_{\xi}$ varying with $[\phi]$ in $\mathbb{P}(W_{\xi})$. This point of view seems to be quite fruitful since it leads to uncovering 'hidden' structures of the filtrations (1.1) such as quivers, Fano toric varieties, dimer models. This in turn provides intriguing connections with such topics as quantum-type invariants, elliptic curves, number theory, mirror symmetry. Perhaps the unifying theme of all those connections is the *higher categorical structures* of the IVHS. At this stage we do not attempt to address this possibility formally, however, various aspects of the refinement presented here strongly point in this direction. The main objective of this writing is to explain

- the constructions of the refinement of IVHS,
- its connections with classical topics in the curve theory,
- how all of the above mentioned 'hidden' structures arise,
- indicate how some of the topics listed above connect with the refinement of IVHS.

In the rest of the paper we give a brief overview of the results and constructions involved and sketch the logical connections between them. Hopefully, this will provide a guide for the reader to browse through the sections of the full text of the forthcoming paper, [R5].

## § 2 Algebraic Kähler structure of $W_\xi$ and the quiver attached to it

The filtrations of $W_\xi$ in (1.1) attached to points $([\xi],[\phi])$ in the cohomological incidence correspondence $P$ in (1.3) arise from a certain linear map

$$\alpha^{(2)}_\xi : \bigwedge^2 W_\xi \longrightarrow H^0(\mathcal{O}_C(K_C)) \tag{2.1}$$

naturally attached to the pair $([\xi],W_\xi)$, a two-form with values in $H^0(\mathcal{O}_C(K_C))$. This turns out to be only part of the structure. Using the Hodge metric on $H^0(\mathcal{O}_C(K_C))$ we can break $W_\xi$ into orthogonal pieces, that is, there is an intrinsically defined orthogonal decomposition

$$W_\xi = \bigoplus_{r=0}^{l_\xi([\phi])} P^r([\xi],[\phi]), \tag{2.2}$$

where $l_\xi([\phi])$ is the length of the filtration in (1.1). This decomposition is related to the filtration by the formula

$$W^s_\xi([\phi]) = \bigoplus_{r=s}^{l_\xi([\phi])} P^r([\xi],[\phi]).$$

In addition to the decomposition, the linear map (2.1) gives rise to the linear map

$$\alpha^{(2)}_\xi(\phi,\bullet) : W_\xi \longrightarrow H^0(\mathcal{O}_C(K_C)).$$

The last summand $P^{l_\xi([\phi])}([\xi],[\phi])$ of the decomposition turns out to be $\alpha^{(2)}_\xi(\phi,\bullet)$-invariant. More precisely, we have the endomorphism

$$\widehat{\alpha^{(2)}}_\xi(\phi,\bullet) : P^{l_\xi([\phi])}([\xi],[\phi])/\mathbf{C}\phi \longrightarrow P^{l_\xi([\phi])}([\xi],[\phi])/\mathbf{C}\phi. \tag{2.3}$$

The study of this endomorphism underlies the assertion about the linear system in Theorem 1.1: we establish a precise dictionary between the eigen spaces of the endomorphism and certain special linear systems on $C$

$$\{\textit{the spectrum of } \widehat{\alpha^{(2)}}_\xi(\phi,\bullet)\} \longrightarrow \{\textit{special line bundles on } C\}. \tag{2.4}$$

Thus the last piece of our filtration establishes links, on the one hand, to classical problems in curve theory such as special divisors, and on the other hand, to Brill-Noether loci in the moduli space of curves.

We now turn to other summands of the decomposition (2.2). Every summand $P^s([\xi],[\phi])$, for $s \in [0, l_\xi([\phi])-1]$, is sent by $\alpha^{(2)}_\xi(\phi,\bullet)$ to the direct sum

$$\bigoplus_{r=s-1}^{l_\xi([\phi])} P^r([\xi],[\phi]),$$

and where the summand $P^{-1}([\xi],[\phi])$ is understood as the orthogonal complement $W^\perp_\xi$ of $W_\xi$ in $H^0(\mathcal{O}_C(K_C))$ with respect to the Hodge metric. Observe that this means that the step

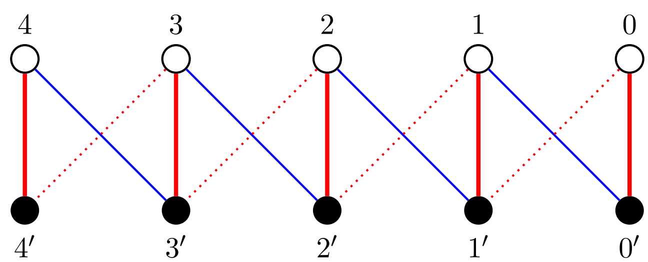
$W^s_\xi([\phi])$ of the filtration is sent to the step $W^{s-1}_\xi([\phi])$, for every $s \geq 1$, a sort of Griffiths transversality property in the classical variation of Hodge structure.

For the integers $r, s$ in $[0, l_\xi([\phi]) - 1]$, denote by $\alpha^{r,s}$ the block of $\alpha^{(2)}_\xi(\phi, \bullet)$ mapping the summand $P^s([\xi], [\phi])$ to $P^r([\xi], [\phi])$. We encode part of these data in the following graph:

• for every summand $P^r([\xi], [\phi])$, $r$ in the range of $[0, l_\xi([\phi]) - 1]$ of the orthogonal decomposition of $W_\xi$, we draw two vertices on the same vertical level; color the top one white $\circ$, and the bottom one black $\bullet$; place the labels $r$ at the white (top) vertex and the label $r'$ for the black (bottom) in the decreasing order from left to right for $r \in [0, l_\xi([\phi]) - 1]$;

• the edges of the graph are drawn between the vertices of different colors only; this is done according to the rule: the top vertex $r$ is connected to the bottom $r'$ and its immediate neighbors $(r-1)'$ and $(r+1)'$.

The resulting graph is denoted $PG_{l_\xi([\phi])}$. The following drawing illustrates the graph $PG_5$, for $l_\xi([\phi]) = 5$:




By orienting the edges from 'white' to 'black' we think of $PG_{l_\xi([\phi])}$ as a quiver. It should be clear, that the graph is constructed so that the orthogonal decomposition of $W_\xi$ in (2.2) together with the maps $\alpha^{r,s}$ provide a representation of the quiver $PG_{l_\xi([\phi])}$. Namely, we place the summand $P^r([\xi], [\phi])$ at the vertices $r$ and $r'$, for every $r$ in the range $[0, l_\xi([\phi]) - 1]$; the vertical edges of the graph are decorated with the endomorphisms

$$\alpha^{r,r} : P^r([\xi], [\phi]) \longrightarrow P^r([\xi], [\phi]),$$

the edges $(r) \to (r-1)'$ are decorated with the maps

$$\alpha^{r,r-1} : P^r([\xi], [\phi]) \longrightarrow P^{r-1}([\xi], [\phi]),$$

for every $r \in [1, l_\xi([\phi]) - 1]$, and the edges $r \to (r+1)'$ with the maps

$$\alpha^{r,r+1} : P^r([\xi], [\phi]) \longrightarrow P^{r+1}([\xi], [\phi]),$$

for every $r \in [0, l_\xi([\phi]) - 2]$.

The following picture emerges:

- the filtrations attached to points of the cohomological incidence correspondence $P$ in (1.3) equip it with the length function

$$\mathfrak{l} : P \longrightarrow \mathbb{N}$$

sending a point $([\xi], [\phi])$ to the length $l_\xi([\phi])$ of the filtration of $W_\xi$ in (1.1); this is a constructible function and it stratifies each stratum $\Sigma^0_r$ of Griffiths stratification into the finer strata according to the values of the length function $\mathfrak{l}$;

- fix a value $l$ in the range of the length function and consider the stratum

$$\mathfrak{L}_l := \mathfrak{l}^{-1}(l);$$

for the rest of the discussion we assume that $l$ is at least 3; from the above it follows:

• the stratum $\mathfrak{L}_l$ comes along with the graph $PG_l$; that graph is bipartite - its vertices are colored white/black and edges connect vertices of different colors only; with the orientation of edges 'white'-to-'black' $PG_l$ is a quiver;

• the points of $\mathfrak{L}_l$ provide quiver representations of $PG_l$ according to the rules discussed above;

• the stratum $\mathfrak{L}_l$ can be decomposed further into the substrata where the dimensions of the graded pieces of the filtration $W^{\bullet}_{\xi}([\phi])$ remain constant; each such substratum is denoted by $\mathfrak{L}_l(h^l, \lambda)$: $h^l$ is the dimension of the last piece of the filtration and $\lambda$ is the partition associated to the graded module

$$W_{\xi}/W^l_{\xi}[\phi] = \bigoplus_{s=1}^{l-1} P^s([\xi],[\phi]).$$

The length strata $\mathfrak{L}_l$ and their substrata $\mathfrak{L}_l(h^l, \lambda)$ provide a geometric part of our refinement of IVHS; the quiver $PG_l$ and $\mathfrak{L}_l$, viewed as a parameter space for representations of $PG_l$, constitute an algebraic part of the refinement.

# § 3 The graph underlying the quiver and the associated Fano toric variety

The graph $PG_l$ is used to introduce another geometric object of our refinement: a Fano toric variety. It can be done formally starting from $PG_l$ as follows: assign the indeterminate $T$ to all vertical edges of the graph, the indeterminates $X_r$ to the edges $(r) \to (r-1)'$, and $Y_r$ to the edges $(r-1) \to (r)'$, for $r \in [1, l-1]$; we think of

$$\{T, X_r, Y_r \,|\, r \in [1, l-1]\}$$

as linear forms on the vector space $V_{2l-1}$ freely generated by the vertices $\{(r), (r)' | \, r \in [1, l-1]\}$ and the symbol $\{e^0\}$ which stands for the vertical edge $(0) \to (0)'$; the form $T$ vanishes on all vertices and $T(e^0) = 1$, the forms $X_r$'s (resp., $Y_r$'s) form the dual basis for the white (resp., black) vertices and vanishes on the black (resp., white) ones. Next we write down the system of quadratic equations

$$X_r Y_r = T^2, \; r = 1, \ldots, l-1;$$

this gives a system of quadrics in $\mathbb{P}(V_{2l-1}) \cong \mathbb{P}^{2(l-1)}$ and we define our variety as the complete intersection of those quadrics; this variety is denoted $\mathbf{H^{1,0}}(PG_l)$ in the main body of the article. The following statement summarizes its properties.

**Proposition 3.1** *The variety $\mathbf{H^{1,0}}(PG_l)$ is a singular Fano toric variety in the projective space $\mathbb{P}(V_{2l-1}) = \mathbb{P}^{2(l-1)}$. It has dimension $(l-1)$ and degree $2^{l-1}$.*

*The hyperplane $T = 0$ intersects $\mathbf{H^{1,0}}(PG_l)$ along the divisor denoted $H_0$. This divisor is the union of $2^{l-1}$ projective subspaces. More precisely, for every subset $A \subset [1, l-1]$ denote by*

*$A^c$ its complement in $[1, l-1]$ and let $\Pi_A$ be the projective subspace of the hyperplane $T = 0$ spanned by the points of $\mathbb{P}(V_{2l-1})$ underlying the vectors*

$$\{(s), (t)' | s \in A,\, t \in A^c\} \subset V_{2l-1};$$

*then we have*

$$H_0 = \bigcup_{A \subset [1,l-1]} \Pi_A,$$

*where the union is taken over all subsets $A$ of $[1, l-1]$.*

The hyperplane sections of $\mathbf{H^{1,0}}(PG_l)$ are (singular) Calabi-Yau varieties. The divisor $H_0$ in the proposition is a particular degenerate case. For the reasons explained in the article [R5], it is called Lagrangian cycle and its irreducible components, the projective spaces $\Pi_A$ in the proposition, could be considered as Lagrangians. We suggest that these should be objects of a Fukaya-type category. This is a subject for a future research. For now we wish to explain the relation of the variety $\mathbf{H^{1,0}}(PG_l)$ to the quiver representation side of the graph $PG_l$.

So far the variety $\mathbf{H^{1,0}}(PG_l)$ and its Lagrangian cycle $H_0$ are defined formally and their relevance to algebraic Kähler structures is not clear. So the appearance of this Fano variety may strike the reader as completely artificial. This is not so, since the quadratic equations defining $\mathbf{H^{1,0}}(PG_l)$ arise from natural deformations of the quiver representations of $PG_l$. Let us briefly explain this.

For this we return to the middle of the incidence diagram (1.3), the projective space $\mathbb{P}(H^0(\mathcal{O}_C(K_C)))$. If we start with a nonzero subspace $W$ in $H^0(\mathcal{O}_C(K_C))$, then the cohomological incidence correspondence $P$ produces the subspace of Kodaira-Spencer classes annihilating $W$:

$$\Xi_W := \{\xi \in H^1(\Theta_C) | \xi\phi = 0, \forall \phi \in W\}.$$

We assume that the dimension of $W$ is at least two and $\Xi_W$ is nonzero. Our algebraic Kähler structure gives linear maps

$$\alpha^{(2)}_\xi(\phi, \bullet) : W_\xi \longrightarrow H^0(\mathcal{O}_C(K_C)).$$

By construction $W$ is contained in every $W_\xi$ as $\xi$ varies in $\Xi_W$. Restricting the maps $\alpha^{(2)}_\xi(\phi, \bullet)$ to $W$, we obtain the map

$$W \longrightarrow (\Xi_W)^* \otimes Hom(W, H^0(\mathcal{O}_C(K_C)))$$

The main point is that this is a sort of Higgs field. More concretely, the operators $\alpha^{(2)}_\xi(\phi, \bullet)$ are commuting in the following sense: for $\xi \neq \xi'$ in $\Xi_W$ we have the equality

$$\alpha^{(2)}_\xi(\phi, \alpha^{(2)}_{\xi'}(\phi, \psi)) \equiv \alpha^{(2)}_{\xi'}(\phi, \alpha^{(2)}_\xi(\phi, \psi)) \mod \mathbf{C}\phi, \tag{3.1}$$

whenever the compositions are defined. This in turn produces commutation relations between the blocks $\{\alpha^{r,s}_\xi(\phi)\}$ and $\{\alpha^{r,s}_{\xi'}(\phi)\}$ of $\alpha^{(2)}_\xi(\phi, \bullet)$ and $\alpha^{(2)}_{\xi'}(\phi, \bullet)$. Now we deform the operators $\alpha^{(2)}_\xi(\phi, \bullet)$ and $\alpha^{(2)}_{\xi'}(\phi, \bullet)$

$$D_\xi(t, \mathbf{x}, \mathbf{y}) = t \sum_{r=0}^{l-1} \alpha^{r,r}_\xi(\phi) + \sum_{r=1}^{l-1} x_r \alpha^{r-1,r}_\xi(\phi) + \sum_{r=0}^{l-2} y_{r+1} \alpha^{r+1,r}_\xi(\phi) + \sum_{s-r \geq 2} \alpha^{s,r}_\xi(\phi),$$

by introducing deformation parameters $\mathbf{x} = (x_r)$ and $\mathbf{y} = (y_r)$ in $\mathbf{C}^{l-1}$ and $t \in \mathbf{C}$, for blocks decorating the edges of the graph $PG_l$. We wish to deform so that the commutativity condition (3.1) would be conserved. The direct computation gives the system of quadratic equations in the formal definition of $\mathbf{H^{1,0}}(PG_l)$. We also gain a conceptual understanding: $\mathbf{H^{1,0}}(PG_l)$ is a parameter space of Higgs fields. This is the reason for the notation alluding to $(1,0)$-forms as well as the name used in the main body of the paper - nonabelian Dolbeault variety.

Though the defining equations of $\mathbf{H^{1,0}}(PG_l)$ come from the deformations of quiver representations arising from our refinement, the variety itself depends on the graph only. It completely forgets our curve $C$. The situation is reminiscent of the classical Jacobian of a curve which takes into account only cohomological data of a curve. The connection with the curve is restored through the Abel-Jacobi maps. Following the analogy with the classical situation, we construct several naive versions of Abel-Jacobi maps:

- the maps from $\mathfrak{L}_l$ to $\mathbf{H^{1,0}}(PG_l) \setminus H_0$, the complement of the Lagrangian cycle,
- the dual of the above assigning to the points of $\mathfrak{L}_l$ hyperplane sections of $\mathbf{H^{1,0}}(PG_l)$,
- for a nonzero section $\phi \in H^0(\mathcal{O}_C(K_C))$ we distinguish the stratum $\mathfrak{L}_l(\phi)$ of $\mathfrak{L}_l$ together with the map into the ring $\mathbf{C}(C)[u, u^{-1}]$ of Laurent polynomials with coefficients in the function field $\mathbf{C}(C)$ of $C$.

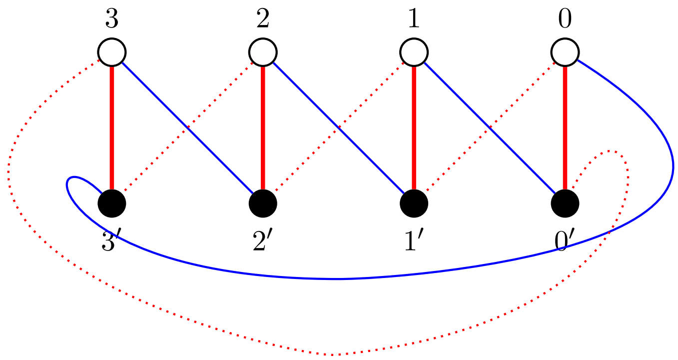
The constructions are given in [R5]. Here we wish to point out the meaning of the second and the third maps. The reader may recall that the hyperplane sections of $\mathbf{H^{1,0}}(PG_l)$ are singular Calabi-Yau varieties, so the second map connects the points of $\mathfrak{L}_l$ with Calabi-Yau varieties, that is, $\mathfrak{L}_l$ becomes a parameter space for families of Calabi-Yau varieties. This could be a source of many interesting invariants supported on $\mathfrak{L}_l$.

For the third map, the reader will need to recall the diagram in (1.3) connecting the cohomological and geometrical incidence correspondences. Conceptually, the third map establishes a connection between the two parts of the diagram: a point $[\phi]$ in the middle space $\mathbb{P}(H^0(\mathcal{O}_C(K_C)))$ of the diagram defines the stratum $\mathfrak{L}_l(\phi)$ in the cohomological incidence $P$ and relates it to the function field of $C$ on the geometrical part of the diagram. The 'strange' appearance of Laurent polynomials will become more natural and interesting once we discuss the next aspect of the refinement.

## § 4 From algebraic Kähler structure of $W_\xi$ to dimer models

We already said that the graphs $PG_l$ associated to the strata $\mathfrak{L}_l$ are bipartite, that is, the vertices come in two different colors (white and black) and each edge of the graph connects vertices of different colors. The graphs of this type are called dimer models and have been intensively studied in recent years in connection with mirror symmetry. There is a vast physics and mathematics literature on the subject. We cite [Go-K] and references therein for more details on the subject.

The graph $PG_l$ is a dimer model with the special property that it is 'almost' trivalent: all vertices have valance 3 except the first and the last pair, that is the vertices with labels $(0), (0)'$ and $(l-1), (l-1)'$. We turn $PG_l$ into a *trivalent dimer model* by connecting the vertex $(0)$ to $(l-1)'$ and the vertex $(l-1)$ to $(0)'$, in accordance to the pattern of edges at the intermediate vertices. The graph obtained this way is denoted $\widehat{PG_l}$. Below is the illustration of the graph $\widehat{PG}_4$.




The graph $\widehat{PG}_l$ is endowed with a structure of a ribbon graph, that is, we choose a cyclic order of edges at every vertex: our choice is counterclockwise, starting with the vertical edge. This ribbon graph structure turns $\widehat{PG}_l$ into a retract of a topological torus $\mathbb{T}$ with $l$ disks removed; the boundary of the disks are so called boundary cycles $B_0, B_1, \ldots, B_{l-1}$ of the ribbon graph $\widehat{PG}_l$; they are labeled by the white vertices. We obtain what in physics literature is called 'brane tilings', the 'tiles' in reference to the polygons formed by the boundary cycles, see [Ken] for physics oriented overview of the subject. In the situation of $\widehat{PG}_l$ all tiles happen to be hexagons; all this is discussed and worked out in [R5].

While the graph $PG_l$ underlies the quiver representations of our IVHS refinement, the completed graph seems to lack this connection. In fact, one of the main features of $\widehat{PG}_l$ is the following extension property.

**Proposition 4.1** *Given a representation $\rho = \{\alpha^{r,s} : P^s \longrightarrow P^s\}$ of the quiver $PG_l$, it can be extended in a canonical way to the representation of the quiver $\widehat{PG}_l$. Those canonical extensions are parametrized by pairs of linear functionals on the space of Laurent polynomials* $\mathbf{C}[q, q^{-1}]$.

The details and consequences of this statement are given in [R5]. Let us discuss the result and its proof informally.

What has been done so far is a finer stratification of the incidence correspondence $\mathbb{P}(\mathcal{W}_{\Sigma^0_r})$ over $\Sigma^0_r$: it is divided into the strata $\mathfrak{L}_l$ where the length of filtrations is fixed and equal to $l$. The graph $PG_l$ appears as a bookkeeping device for arranging those filtrations as quiver representations of $PG_l$. The graph could be viewed as 'open' in a sense that its quiver representation

$$\rho = \{\alpha^{r,s} : P^s \longrightarrow P^r\}$$

has 'ends', the spaces $P^0$ and $P^{l-1}$; those could be viewed as 'output' *objects* of the quiver representations associated to our filtrations; the obvious *morphisms* are the vector spaces

$$P^-([\xi],[\phi]) := Hom_{\mathbf{C}}(P^0([\xi],[\phi]), P^{l-1}([\xi],[\phi])),$$
$$P^+([\xi],[\phi]) := Hom_{\mathbf{C}}(P^{l-1}([\xi],[\phi]), P^0([\xi],[\phi]))$$

between the end objects; to simplify the notation we will often omit the reference to the point $([\xi],[\phi])$ in $\mathfrak{L}_l$, but the reader should keep in mind this variation.

The graph $\widehat{PG_l}$ connects the ends of the graph $PG_l$ combinatorially/topologically by introducing the additional edges. The proposition tells us that we can consistently label those edges using the data of the representations of the 'open' graph and some 'quantum' parameters, the space $Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C})$. Concretely, the proof of the proposition tells us that there are distinguished linear maps

$$\tau^{\pm}: Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C}) \longrightarrow P^{\pm}$$

which allow to connect the 'end' objects of the quiver representations of $PG_l$. Observe the appearance of Laurent polynomials: here they appear naturally as traces of loop versions of certain Lie algebras attached to the quiver representations. The technical part of the construction of $\tau^{\pm}$ involves certain closed paths on $\widehat{PG_l}$ called zig-zag paths, see [R5] for details.

Thus the above maps is an extra structure on the *morphisms* between the end objects of the representations of the quiver $PG_l$. One could wonder if we have morphisms between morphisms and if those have additional structures. In other words, we are asking for *higher categorical structures of the IVHS*. In [R5] we provide indications that this might be the case. There are two different directions: one connects to symplectic geometry and the other to the ideas of the Topological Quantum field theory (TQFT), see [At]. The first stems from the fact that the space

$$P([\xi],[\phi]) = P^{-}([\xi],[\phi]) \oplus P^{+}([\xi],[\phi])$$

combining the morphisms between the end objects is naturally a symplectic vector space: this is due to the fact that the two summands are dual to each other. We construct Lagrangian subspaces in $P([\xi],[\phi])$ which could be viewed as 1-morphism; symplectic morphisms between those Lagrangians could be 2-morphisms and a possible connection to Fukaya categories arises.

The second direction uses the fact that the graph $\widehat{PG_l}$ is a dimer model and as such it comes with the set of perfect matchings - configurations of edges which cover all vertices of the graph and precisely once. We suggest that this set can be categorified and this could provide higher order 'multiplications' in the tensor algebra of the graded space $W_\xi/W^l_\xi([\phi])$, see [R5] for details.

The extension of the representations of $PG_l$ to the ones of the quiver $\widehat{PG_l}$ has a geometric counterpart: the strata $\mathfrak{L}_l$ become related to the moduli spaces $\mathfrak{M}^l_1$ of elliptic curves with $l$ marked points. This relation comes from metrics on the ribbon graph $\widehat{PG_l}$ and then one appeals to the general theory relating metrized ribbon graphs with the moduli spaces of curves with marked points, the theory which goes back to the works of Harer, Mumford, Penner, Thurston, see [Har], [MuP] for overviews. The main point is that a quiver representation of $\widehat{PG_l}$ *determines* a metric on $\widehat{PG_l}$ and this metric turns the topological torus $\mathbb{T}$ into an elliptic curve with $l$ marked points, the barycenters of the boundary cycles of $\widehat{PG_l}$. This is the contents of the following.

**Proposition 4.2** *Given a representation $\rho = \{\alpha^{r,s}: P^s \longrightarrow P^s\}$ of the quiver $PG_l$, it can be extended in a canonical way to the representation of the quiver $\widehat{PG_l}$. Those canonical extensions are parametrized by pairs of linear functionals on the space of Laurent polynomials $\mathbf{C}[q,q^{-1}]$. For every pair $(F,G)$ of such functionals one obtains a continuous map*

$$\wp_l(F,G): \mathfrak{L}_l \longrightarrow \mathfrak{M}^l_1.$$

*In other words we have a continuous map*

$$\wp_l : \left(Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C})\right)^2 \longrightarrow \left(\mathfrak{M}_1^l\right)^{\mathfrak{L}_l}.$$

*which sends a pair $(F,G)$ on the left side to the map $\wp_l(F,G)$.*

The maps $\wp_l(F,G)$ divide $\mathfrak{L}_l$ into substrata, the fibres of $\wp_l(F,G)$, to which is attached a particular elliptic curve with $l$ marked points. One also can phrase the above result as attaching to each point of $\mathfrak{L}_l$ *families of elliptic curves with $l$ marked points,* where the families are parametrized by the space $\left(Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C})\right)^2$. In other words we can rewrite the map $\wp_l$ in Proposition 4.2 as follows

$$\wp_l : \mathfrak{L}_l \longrightarrow \left(\mathfrak{M}_1^l\right)^{\left(Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C})\right)^2}.$$

The graph $\widehat{PG}_l$ has another particular feature: it is endowed with a distinguished automorphism of order $l$. This automorphism is denoted by $\sigma_l$. By choosing a metric on $\widehat{PG}_l$ invariant with respect to $\sigma_l$ allows us to be more precise about the marked points: if $l$ is at least four, the marked points form a cyclic subgroup of order $l$ in the corresponding elliptic curve and the subgroup comes with a particular choice of a generator. In other words, the $\sigma_l$-invariant metrics on $\widehat{PG}_l$ take us to the modular curve $Y_1(l)$, the moduli of pairs $(E,p)$ where $E$ is an elliptic curve and $p$ is a point of order $l$ in $E$. Actually, any metric of $\widehat{PG}_l$ can be made $\sigma_l$-invariant by choosing a triple of standard symmetric polynomials in $l$ indeterminates. It is well-known that the standard symmetric functions, such as monomial, elementary, power functions are parametrized by partitions, see [Mac]. Thus once we specify which symmetric functions we associate to partitions, the $\sigma_l$-invariant metrics of $\widehat{PG}_l$ can be obtained by starting from *any* metric on $\widehat{PG}_l$ and then making it $\sigma_l$-invariant according to a choice of a triple of partitions having at least $l$ parts.

**Proposition 4.3** *Assume $l \geq 4$. For any triple of partitions*

$$\mu = (\mu_0, \mu_+, \mu_-)$$

*with at least $l$ parts, the map $\wp_l$ in Proposition 4.2 gives rise to the continuous map*

$$\wp_{l,\mu} : \mathfrak{L}_l \longrightarrow (Y_1(l))^{\left(Hom_{\mathbf{C}}(\mathbf{C}[q,q^{-1}],\mathbf{C})\right)^2}.$$

The reader may recall that the strata $\mathfrak{L}_l$ are further stratified into the substrata $\mathfrak{L}_l(h^l,\lambda)$ where the labels $(h^l,\lambda)$ are respectively the dimension of the $l$-th piece of filtration and $\lambda$ is the partition associated to the dimensions of the graded pieces of the quotient $W_\xi / W_\xi([\phi])$. In addition, the partition $\lambda'$ conjugate to $\lambda$ has exactly $l$ parts. Thus the substrata $\mathfrak{L}_l(h^l,\lambda)$ become *canonical labels* for the maps of $\mathfrak{L}_l$ into the modular curve $Y_1(l)$

**Proposition 4.4** *Assume $l \geq 4$. Then each substratum $\mathfrak{L}_l(h^l,\lambda)$ of $\mathfrak{L}_l$ gives rise to a distinguished continuous map*

$$\wp_{l,\lambda} : \mathfrak{L}_l \longrightarrow Y_1(l).$$

The connection of the IVHS refinement with the moduli spaces of the elliptic curves suggests other possibilities of categorifications as well as links to number theory, automorphic forms.

## § 5 Further research direction and applications

There are several obvious directions to explore:

• to apply the refinement constructions to study families of curves; in particular, to apply them to study surfaces with nonhyperelliptic fibrations and, more generally, smooth projective varieties fibred by nonhyperelliptic curves of genus $g$,

• to develop IVHS refinement for higher dimensional varieties, in particular, compact complex manifolds with the canonical bundle very ample,

• to study the categorical aspects of the IVHS refinement.

As we alluded to at various points of the introduction, the connections with number theory emerge: this comes from the possibility to label the extensions of quiver representations in Proposition 4.1 by arithmetic objects coming from number fields or from the fields of functions of curves over finite fields, see [R5] for more details.

Our refinement of IVHS naturally connects to quivers, Fano toric varieties, dimers on a trivalent graphs. Those topics figure prominently in actual research in connection with mirror symmetry. The following is a speculative perspective on our constructions from the point of view of mirror symmetry .

## § 6 The mirror symmetry perspective

Let us go back to the diagram (1.3). As we explained earlier in the introduction, our refinement consists of attaching to the IVHS pair $([\xi], W_\xi)$ on the cohomological side of the diagram the filtrations of $W_\xi$ varying holomorphically with $[\phi] \in \mathbb{P}(W_\xi)$. It was also alluded to that this could be envisaged as a sort of Kähler structure varying with $[\phi]$. Part of this structure, according to the second item of Theorem 1.1, 'counts' certain rational normal curves in the projective space $\mathbb{P}(H^0(\mathcal{O}_C(K_C))^*)$, the bottom right of the diagram (1.3). This invokes some of the ingredients of Mirror symmetry: the objects, say Calabi-Yau varieties, are equipped with two structures - holomorphic and symplectic (B-side and A-side in physics parlor); for two mirror objects $M$ and $M'$, the mirror symmetry predicts 'isomorphisms' between sides of different nature, that is, the B-side (resp. A-side) of $M$ is 'isomorphic' to the A-side (resp. B-side) of $M'$. From this perspective our suggestion is

•*every part of the diagram* (1.3) *should come with its A- and B- sides;*
•*the adjacent parts of the diagram are related by functorial equivalences* (6.1)
*between the sides of different nature.*

For example, on the bottom left, $\mathbb{P}(H^1(\Theta_C))$ is the projectivization of the space of the infinitesimal variation of complex structures on $C$; this qualifies for the B-side; our refinement could be viewed as algebraic Kähler structures along the arrow $p_1$ of the diagram and this is (a part of) A-side. This A-side should be functorially related to the B-side in the middle of the diagram and that one to the A-side on the bottom right. Rational normal curves in $\mathbb{P}(H^0(\mathcal{O}_C(K_C))^*)$ related to the canonical embedding of $C$ should qualify for (a part of) the A-side of the canonical embedding $C \subset \mathbb{P}(H^0(\mathcal{O}_C(K_C))^*)$. The second item of Theorem 1.1 relating algebraic Kähler structures on $\mathbb{P}(H^1(\Theta_C))$ to the rational normal curves related to the canonical embedding of $C$ could be viewed as an example of the hypothetical functorial relations in (6.1). The Hilbert polynomial

$$H_{[\xi],[\phi]}(q) = \sum_{i=0}^{l} dim \left( W^i_{\xi} / W^{i+1}_{\xi} \right) q^i$$

of the filtration (1.1) could be viewed as a numerical output of this functorial relation. According to Theorem 1.1, the function counts the dimensions of families of certain rational curves related to the canonical embedding of $C$, a sort of Gromov-Witten type invariant.

The homological Mirror symmetry conjecture of Kontsevich recasts Mirror symmetry as equivalences between pairs of categories attached to each of the mirror partners, [K]. In the case of Calabi-Yau varieties the two categories are the derived category of coherent sheaves, the holomorphic or B-side, and Fukaya category, the symplectic or A-side; for mirror partners $M$ and $M'$ the conjecture predicts the equivalences of

- the derived category of $M$ with the derived Fukaya category of $M'$,
- the derived Fukaya category of $M$ with the derived category of $M'$.

In view of the homological mirror symmetry conjecture of Kontsevich the suggestion (6.1) can be restated as follows:

$$\begin{gathered} \bullet\textit{every part of the diagram } (1.3) \textit{ should come with two types of categories:} \\ \textit{one related to the derived category of coherent sheaves,} \\ \textit{the other to the Fukaya category of some (auxiliary) varieties;} \\ \bullet\textit{the adjacent parts of the diagram are related by equivalences} \\ \textit{between the categories of different type.} \end{gathered} \tag{6.2}$$

Let us see how our situation fits into this pattern. The starting point for our constructions is to view the Kodaira-Spencer classes $\xi$ in $H^1(\Theta_C)$ as the extension classes via the identification

$$H^1(\Theta_C) = H^1(\mathcal{O}_C(-K_C)) \cong Ext^1(\mathcal{O}_C(K_C), \mathcal{O}_C).$$

In other words, a nonzero Kodaira-Spencer class $\xi$ is viewed as the corresponding extension sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_\xi \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0. \tag{6.3}$$

This way the B-side of $\mathbb{P}(H^1(\Theta_C))$ becomes related to the category of (short) exact complexes (up to the $\mathbf{C}^\times$-action by the multiplication on the arrows of the complex) on $C$. Furthermore, on the cohomology level we have the exact complex

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{E}_\xi) \longrightarrow H^0(\mathcal{O}_C(K_C)) \xrightarrow{\xi} H^1(\mathcal{O}_C),$$

where the rightmost arrow is the cup product with $\xi$. In particular, if the rank of the map

$$H^0(\mathcal{O}_C(K_C)) \xrightarrow{\xi} H^1(\mathcal{O}_C)$$

is $r$, that is the point $[\xi]$ lies in the open stratum

$$\Sigma^0_r := \Sigma_r \setminus \Sigma_{r-1},$$

then the rank 2 bundle $\mathcal{E}_\xi$ in the middle of the extension sequence has the property

$$h^0(\mathcal{E}_\xi) = g - r + 1.$$

More precisely, our space

$$W_\xi = ker\Big(H^0(\mathcal{O}_C(K_C)) \xrightarrow{\xi} H^1(\mathcal{O}_C)\Big),$$

is a part of the following exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{E}_\xi) \longrightarrow W_\xi \longrightarrow 0. \tag{6.4}$$

The pair of two exact sequences

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E}_\xi \longrightarrow \mathcal{O}_C(K_C) \longrightarrow 0, \tag{6.5}$$

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{E}_\xi) \longrightarrow W_\xi \longrightarrow 0,$$

could be viewed as a lifting of the IVHS data

$$([\xi], W_\xi)$$

to the category of perfect complexes on $C$ with an extra data on the level of global sections. Thus we have a natural way to relate $\mathbb{P}(H^1(\Theta_C))$ equipped with the Griffiths stratification and the arrow $p_1$ of the diagram (1.3) to the derived category of coherent sheaves on $C$.

The categorical data (6.5) has an additional structure: the space of the global sections $H^0(\mathcal{E}_\xi)$ comes along with the exterior product

$$\bigwedge^2 H^0(\mathcal{E}_\xi) \longrightarrow H^0(\bigwedge^2 \mathcal{E}_\xi) = H^0(\mathcal{O}_C(K_C)), \tag{6.6}$$

since the extension sequence (6.3) implies $\bigwedge^2 \mathcal{E}_\xi = \mathcal{O}_C(K_C)$. This could be viewed as an $H^0(\mathcal{O}_C(K_C))$-valued two-form attached to $([\xi], W_\xi)$. The above exterior product is related to the one we have seen in (2.1). It is an essential ingredient in constructing the filtration (1.1). We suggest to view it as a sort of 'symplectic' structure attached to $([\xi], W_\xi)$. This should eventually lead to Fukaya-type categories related to the left side of the diagram (1.3).

From the discussion on the nonabelian Dolbeault space $\mathbf{H^{1,0}}(PG_l)$ as the parameter space of Higgs structures we also have a glimpse at the categories in the middle of the diagram (1.3): the objects are projective subspaces $\mathbb{P}(W)$ in $\mathbb{P}(H^0(\mathcal{O}_C(K_C)))$ with the subspace $\Xi_W$ of $H^1(\Theta_C)$ annihilating $W$ - the data delivered by the cohomological correspondence $P$. Over $\mathbb{P}(H^0(\mathcal{O}_C(K_C)))$ we also have the geometric correspondence

$$q_1 : \mathcal{Z}_C \longrightarrow \mathbb{P}(H^0(\mathcal{O}_C(K_C))).$$

On this side the basic sheaf object is the direct image

$$(q_1)_*(\mathcal{O}_{\mathcal{Z}_C}) = (q_1)_*(q_2^*\mathcal{O}_C)$$

of the structure sheaf $\mathcal{O}_{\mathcal{Z}_C}$ of $\mathcal{Z}_C$. This is locally free with the fibre over a point $[\phi] \in \mathbb{P}(H^0(\mathcal{O}_C(K_C)))$ being the space

$$H^0(\mathcal{O}_{Z_\phi})$$

of functions on the divisor $Z_\phi = (\phi = 0)$, the zero locus of $\phi$. The fibre of the projection

$$p_2 : P \longrightarrow \mathbb{P}(H^0(\mathcal{O}_C(K_C)))$$

in (1.3) over $[\phi]$ is the projectivization of the kernel $\Xi_\phi$ of the map

$$H^1(\Theta_C) = H^1(\mathcal{O}_C(-K_C)) \xrightarrow{\phi} H^1(\mathcal{O}_C)$$

and we can see its relation to the space $H^0(\mathcal{O}_{Z_\phi})$. Namely, the above multiplication by $\phi$ on the cohomology level comes from the exact sequence of sheaves on $C$

$$0 \longrightarrow \mathcal{O}_C(-K_C) \xrightarrow{\phi} \mathcal{O}_C \longrightarrow \mathcal{O}_{Z_\phi} \longrightarrow 0.$$

This gives rise to the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_{Z_\phi}) \longrightarrow \Xi_\phi \longrightarrow 0.$$

The subspace $\Xi_W$ is contained in $\Xi_\phi$ for all nonzero $\phi \in W$. Lifting this subspace to $H^0(\mathcal{O}_{Z_\phi})$ gives a distinguished subspace $\widetilde{\Xi}_W(\phi)$ of functions on $Z_\phi$ containing the subspace of constant functions. As $[\phi]$ varies in $\mathbb{P}(W)$ the subspaces $\widetilde{\Xi}_W(\phi)$ fit together to form a subsheaf, call it $\mathcal{X}_W$, of the restriction of the sheaf $(q_1)_*(\mathcal{O}_{\mathcal{Z}_C})$ to $\mathbb{P}(W)$. Thus the cohomological pair $(\mathbb{P}(W), \Xi_W)$ lifts to the level of coherent sheaves supported on $\mathbb{P}(W)$, the pair $(\mathbb{P}(W), \mathcal{X}_W)$:

$$(\mathbb{P}(W), \Xi_W) \rightsquigarrow (\mathbb{P}(W), \mathcal{X}_W).$$

We can do more: the subspaces $\widetilde{\Xi}_W(\phi)$ produce the filtrations of $H^0(\mathcal{O}_{Z_\phi})$ which are cousins of the filtrations (1.1) on the side of $\mathbb{P}(H^1(\Theta_C))$. This in turn leads to the graphs of the *same* type as $PG_l$ as well as to its own nonabelian Dolbeault varieties with their Lagrangian cycles! The suggestion (6.2) gains somewhat in substance.

At this stage the suggestion formulated in (6.2) serves as a working hypothesis. However, by following it, we believe that one can discover interesting new structures in the theory of curves and beyond.

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