# CONVOLUTION IDENTITIES FOR DIVISOR FUNCTIONS TALK LILLE MAY 2025

KSENIA FEDOSOVA
(BASED ON WORK WITH KIM KLINGER-LOGAN, AND DANYLO RADCHENKO)

**Special cases of main theorems.** In this talk, we will consider convolution sums of divisor functions. And instead of diving in and giving explicit statements of theorems, I will start by giving you special examples of what we will be able to prove.

We denote $\sigma_a(n) = \sum_{d|n, d>0} d^a$. For examples, $\sigma_0(-2) = 1^0 + 2^0 = 2$.

- **Example 1.** For $n \in \mathbb{N}$, and $\varphi_1(n_1, n_2) = 2 - \frac{n_2 - n_1}{n_2 + n_1} \log\left|\frac{n_2}{n_1}\right|$, we have
$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\} \\ n_1 + n_2 = n}} \varphi_1(n_1, n_2)\sigma_0(n_1)\sigma_0(n_2) = \big(2 - \log\big(4\pi^2|n|\big)\big)\sigma_0(n),$$
- **Example 2.** (conjectured in [1])
For $n \in \mathbb{N}$ and
$$\varphi_2(n_1, n_2) = -\frac{1}{4}\Big(\frac{n_1^2}{n_2^2} + \frac{n_2^2}{n_1^2}\Big) - \frac{7}{2}\Big(\frac{n_1}{n_2} + \frac{n_2}{n_1}\Big) + \frac{47}{2} + 15\left(\frac{n_2 - n_1}{n_1 + n_2}\right)\log\left|\frac{n_1}{n_2}\right|,$$
we have
$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}\setminus\{0\} \\ n_1 + n_2 = n}} \varphi_2(n_1, n_2)\sigma_2(n_1)\sigma_2(n_2) = \left(\frac{\zeta(2)n^2}{2} + 30\zeta'(-2)\right)\sigma_2(n),$$
where $\zeta$ denotes the Riemann zeta function.
- **Example 3.** For $n \in \mathbb{N}$ and
$$\varphi_3(n_1, n_2) = (n_1 - 3n_2)|n_1|^{\frac{1}{2}} + (n_2 - 3n_1)|n_2|^{\frac{1}{2}},$$
we have
$$\sum_{\substack{n_1 \in \mathbb{Z}\setminus\{0, n\} \\ n_1 + n_2 = n}} \varphi_3(n_1, n_2)\sigma_{-\frac{1}{2}}(n_1)\sigma_{-\frac{1}{2}}(n_2) = -\sigma_{-\frac{1}{2}}(n)\left(2n^{\frac{3}{2}}\zeta(\tfrac{1}{2}) + \frac{3}{2\pi}n\zeta(\tfrac{3}{2})\right).$$

Note:

(1) there are infinitely many summands in each sum,
(2) all sums considered are absolutely convergent; it is not a meromorphic continuation,

(3) we took $n > 0$ for the convenience; the case $n < 0$ follows immediately, and the case $n = 0$ is completely described by the Ramanujan formula

$$\sum_{n_1 \in \mathbb{N}} \frac{\sigma_{r_1}(n_1)\sigma_{r_2}(n_1)}{n_1^s} = \frac{\zeta(s)\zeta(s-r_1)\zeta(s-r_2)\zeta(s-(r_1+r_2))}{\zeta(2s-(r_1+r_2))}.$$

**Related results.**

- finite sums, odd divisor functions, $M \in \mathbb{N}$.
  - 1804-1851 attributed to Jacobi,

$$\sum_{\substack{1 \le n_1 \le n-1 \\ n_1+n_2=n}} \sigma_3(n_1)\sigma_3(n_2) = \frac{\sigma_7(n) - \sigma_3(n)}{120}.$$

  - [2] introduced method of calculations of

$$\sum_{\substack{1 \le n_1 \le n-1 \\ n_1+n_2=n}} \sigma_{r_1}(n_1)\sigma_{r_2}(n_2)n_1^j, \quad j \in \mathbb{N}_0, r_1, r_2 \in 2\mathbb{N}_0 + 1$$

- [9] showed for any $C^\infty$-function with compact support,

$$\sum_{1 \le m_1 \le \infty, m_1+m_2=m} \sigma_0(m_1)\sigma_0(m_2) W\left(\frac{m_1}{m_1+m_2}\right)$$

  is explicitly given; involves Maass wave forms and integration against Riemann zeta function. Despite that it is given by an infinite sum, the sum is, in fact, finite.
- [8] informal proof of Example 2. Involves non-justifiable interchanging of limits.
- [5] generalization of [8].

## 2. Motivation

Let $f : \mathbb{H} \to \mathbb{C}$ be a function satisfying the equation

$$(\Delta - 12)f(z) = -E_{3/2}^2(z), \quad z \in \mathbb{H}, \quad \Delta = y^2(d^2/dx^2 + d^2/dy^2).$$

Using separation of variables, we obtain that $n$-th Fourier term of $f$ for $n \neq 0$ satisfies a certain ODE that I will not write and thus belongs to a 2-dimensional space[1]

(2.1)

$$\hat{f}_n \in \Big\{ \alpha_n \sqrt{y} K_{1/2}(|n|y) + \beta_n \sqrt{y} I_{1/2}(|n|y) \\ + \sum_{\substack{n_1,n_2 \in \mathbb{Z}\setminus\{0\} \\ n_1+n_2=n}} \sum_{i,j=0}^{1} \sum_{k=0}^{2r-1} q_{i,j,k}(n_1,n_2,y) y^{1-k} K_i(|n_1|y) K_j(|n_2|y) \,|\, \alpha_n, \beta_n \in \mathbb{C} \Big\},$$

where $q_{i,j,k}$ is some function in $n_1$ and $n_2$.

If we demand that the function grows not too fast in cusps, then $\beta_n = 0$.

[1]The equation below is slightly lying: there are summands missing corresponding to the cases $n_1 = n, n_2 = 0$ and $n_1 = 0, n_2 = n$.

- There exists an algorithm to calculate $q_{i,j,k}$ for Fourier coefficients of
$$(\Delta - r(r+1))f(z) = E_{a+1/2}(z)E_{b+1/2}(z)$$
for physically relevant $r, a, b \in \mathbb{N}_0$ [4] and [3].

Naively interchanging finding asymptotic expansion and an infinite sum and assuming Example 2 holds, every function from the space has asymptotic expansion
$$\hat{f}_n(z) = \alpha_n y^{-3} + ..., \quad y \to 0.$$
On the other hand, *physical condition, the weak coupling limit*
$$f(z) = O(y^3), \quad y \to \infty.$$
From automorphy[2], $\alpha_n = 0$.

## 3. Main theorem (case of even divisors)

Let $Q_d^{(\alpha,\beta)}(x)$ be the *Jacobi function of the second kind* defined as follows:
$$Q_d^{(\alpha,\beta)}(x) = \frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2^{d+1}} \int_{-1}^{1} \frac{(1-t)^{d+\alpha}(1+t)^{d+\beta}dt}{(x-t)^{d+1}}\,, \qquad x \in \mathbb{C} \setminus [-1,1]\,.$$
Examples:

- $Q_1^{(0,0)}(x) = \frac{x}{2}\log|\frac{x+1}{x-1}| - 1$.
- for $\alpha, \beta, d \in \mathbb{Z}_{\geq 0}$,
$$Q_d^{(\alpha,\beta)}(x) = \frac{(-1)^\alpha}{2} P_d^{(\alpha,\beta)}(x) \log\left|\frac{x+1}{x-1}\right| + \frac{R(x)}{(x-1)^\alpha(x+1)^\beta},$$
and $R \in \mathbb{Q}[x]$ is a polynomial of degree $d + \alpha + \beta - 1$.

**Lemma 3.1.** [6] *For any $d \geq 1$, $r_1, r_2 \in 2\mathbb{Z}_{>0}$ and $n \in \mathbb{N}$,*
(3.1)
$$\sum_{\substack{n_1,n_2 \in \mathbb{Z}\setminus\{0\}\\ n_1+n_2=n}} Q_d^{(r_1,r_2)}\left(\frac{n_2-n_1}{n_1+n_2}\right)\sigma_{r_1}(n_1)\sigma_{r_2}(n_2) = (-1)^d C_d^{(r_1,r_2)}(n)\sigma_{r_1}(n) - C_d^{(r_2,r_1)}(n)\sigma_{r_2}(n) + \frac{a_n}{n^d}\,,$$
*where*
$$C_d^{(r_1,r_2)}(n) = \frac{(r_2-1)!(r_1+d)!}{2(r_1+r_2+d)!}\zeta(r_2)n^{r_2} + \binom{d+r_2}{d}\frac{\zeta'(-r_2)}{2}.$$
*where $a_n$ is the $n$-th Fourier coefficient of a cusp form of weight*
$$k := 2d + r_1 + r_2 + 2$$

[2]By [7], if $h(x+iy)$ is an $SL(2,\mathbb{Z})$-invariant function on the upper half plane satisfying the large-y growth condition $h(x+iy) = O(y^s)$ for some $s > 1$, then each Fourier mode $\widehat{h}_n(y) = \int_0^1 h(x+iy)e^{-2\pi i n x}dx$ of $h$ satisfies the bound $\widehat{h}_n(y) = O\big(y^{1-s}\big)$ for small $y$. In particular, the small-y boundary condition for any mode number $n$ is
$$\widehat{f}_n(y) = O\big(y^{-2}\big).$$

*on* $\mathrm{SL}_2(\mathbb{Z})$*, given by* $h = \sum_f \lambda_f f$*, where* $f$ *runs over normalized Hecke eigenforms*[3] *of weight* $k$ *and level* $1$*, and*

$$\lambda_f = \frac{\pi(-1)^{d+r_2/2+1}}{2^k}\binom{k-2}{d}\frac{L^\star(f,d+1)L^\star(f,r_1+d+1)}{\langle f,f\rangle}.$$

*Above* $L^\star(f,\cdot)$ *is the completed* $L$*-function of* $f$*:*

$$L^\star(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s).$$

Remarks to the lemma:

- in Examples 1 and 2, the cusp forms are of weights less than 8.
- $r_1, r_2 \in \mathbb{Z}_0$ is also allowed, but formulas become longer,
- informal proof introduced by Miller, Radchenko, Klinder-Logan in [8] and extended by F., Klinder-Logan in [5] allows to calculate the first two terms of the sum
- the formula in the lemma allows to calculate any convergent expression of the form

$$\sum \varphi(n_1,n_2)\sigma_{r_1}(n_1)\sigma_{r_2}(n_2)$$

  for any

$$\varphi(n_1,n_2) := \sum_{j=-r_1}^{d-1} A_j n_1^j + \sum_{j=-r_2}^{d-1} B_j n_2^j + \sum_{j=0}^{d}\big(C_j n_1^j \log|n_1| + D_j n_2^j \log|n_2|\big) \tag{3.2}$$

- $L$-values are evaluated at their critical points,
- a certain combination of generalized Eisenstein series has no cuspidal components; regularized large $N$ expansion of certain integrated correlators in $SU(N)$ $\mathcal{N}=4$ SYM theory.

## Sketch of the proof of the main theorem

The main idea to use generating functions. Proving this equality is to consider it to be the $n$-th Fourier coefficient of a certain cusp form. More precisely, we would like to find some $\Phi$ of weight $k$, and then write

$$\Phi(\tau) = \sum_{n=1}^{\infty} a_n q^n = \sum_{f \text{ is a Hecke eigenform of weight } k} \frac{\langle f,\Phi\rangle}{\langle f,f\rangle} f.$$

Then our convolution identity should read "$n$-th Fourier coefficient of l.h.s. is the $n$-th Fourier coefficient of a r.h.s.".

We will obtain $\Phi$ from a so-called *holomorphic projection*[4] of a certain automorphic function. The next lemma is about that one can project any automorphic form to a cusp form; and then relate their Fourier coefficients.

[3] We say that a Hecke eigenform is normalized if its first non-zero Fourier coefficient is equal to 1.
[4]

**Lemma 3.2** (Holomorphic projection lemma)**.** *Let* $\widetilde{\Phi}$ *be a non-holomorphic modular form of weight* $k > 2$ *for* $\mathrm{SL}_2(\mathbb{Z})$ *with a Fourier expansion* $\widetilde{\Phi}(z) = \sum_{m\in\mathbb{Z}} a_m(y)e^{2\pi i m x}$*, and suppose that for some*

We take non-holomorphic Eisenstein series of weight $2q$ and spectral parameter $s$:

$$y^{-q}E^*_{2q}(\tau,s)=\theta_q(s)\sum_{\gamma\in\Gamma\backslash\Gamma_\infty}\frac{\Im(\gamma.z)^{s-q/2}}{(c\tau+d)^q},$$

where $\theta_q(s)=\pi^{-s}\Gamma(s+q)\zeta(2s)$. It is automorphic with weight $2q$. Moreover, it has the Fourier expansion

$$E^*_{2q}(\tau,s)=\theta_q(s)y^s+\theta_q(1-s)y^{1-s}+(-1)^q\sum_{n\neq 0}\frac{\sigma_{2s-1}(n)}{|n|^s}W_{q,s-1/2}(4\pi ny)e^{2\pi inx},$$

where $W_{q,s-1/2}$ is the Whittaker function. Further we take

$$\widetilde{\Phi}(\tau)=E^*_{2k_1}(\tau,m_1+1/2)E^*_{2k_2}(\tau,m_2+1/2)y^{-k_1-k_2},$$

where we choose the integers $k_1$ and $k_2$ to satisfy $2k_1+2k_2=k=r_1+r_2+2d+2$ and $k_i\geq m_i$. It satisfies the properties of holomorphic projection lemma. The $n$-th Fourier coefficient of $\widetilde{\Phi}(\tau)$ can be calculated as

$$a_n(y)=\sum_{n_1+n_2=n}a_{n_1,n_2}(y)$$

where

$$a_{n_1,n_2}(y)=\frac{(-1)^{k_1+k_2}\sigma_{r_1}(n_1)\sigma_{r_2}(n_2)}{|n_1|^{r_1/2+1/2}|n_2|^{r_2/2+1/2}}W_{k_1,m_1}(4\pi n_1y)W_{k_2,m_2}(4\pi n_2y)y^{k_1+k_2-2}.$$

Now, we want to use the holomorphic projection lemma for $\tilde{\Phi}$. We find that

$$\Phi=\sum_{n=1}^{\infty}a_nq^n=\sum_{n=1}^{\infty}\sum_{n_1+n_2=n}a_{n_1,n_2}q^n,$$

and, up to a constant (we use the formula in the holomorphic projection lemma and calculate all the integrals),

$$a_{n_1,n_2}=\sigma_{r_1}(n_1)\sigma_{r_2}(n_2)Q_d^{(r_1,r_2)}(\tfrac{n_2-n_1}{n_2+n_1}),$$

and

$$a_{n,0}=(-1)^dC_d^{(r_1,r_2)}(n)\sigma_{r_1}(n),\quad a_{0,n}=C_d^{(r_2,r_1)}(n)\sigma_{r_2}(n).$$

Thankfully, Diamantis and O'Sullivan already calculated the Petersson inner product:

$$\langle f,\Phi\rangle\overset{\text{HP}}{=}\langle f,\widetilde{\Phi}\rangle=2(-1)^{k_2}\pi^kL^*(f,d+1)L^*(f,r_2+d+1),$$

that finishes the proof.

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*$\varepsilon>0$ we have $\widetilde{\Phi}(z)=O(y^{-\varepsilon})$ as $z\to i\infty$. Define*

$$a_m=\frac{(4\pi m)^{k-1}}{(k-2)!}\int_0^\infty a_m(y)e^{-2\pi my}y^{k-2}dy,\quad m>0.$$

*Then the function $\Phi(z)=\sum_{m>0}a_me^{2\pi imz}$ is a holomorphic cusp form of weight $k$ for $\mathrm{SL}_2(\mathbb{Z})$ and moreover $\langle f,\Phi\rangle=\langle f,\widetilde{\Phi}\rangle$ for all $f\in S_k(\mathrm{SL}_2(\mathbb{Z}))$.* □

## References


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