Motives for Feynman integrals

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based on with Leonardo de la Cruz, Charles Doran, Andrew Harder, Pierre Lairez, Eric Pichon-Pharabod,

and work in progress with Leonardo de la Cruz, and Pavel Novichkov

Feynman integrals are ubitquitous to many problems in physics ranging from elementary particle physics to classical gravity computations
The generic form of a Feynman integral is

$$I_\Gamma(\underline{z}) = \int_{\Delta_n} \omega_\Gamma(\underline{z}, \underline{x})\, \Omega_0$$

with the domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0, \dots, x_n \geqslant 0 | [x_1, \dots, x_n] \in \mathbb{P}^{n-1}\}$$

Ω_0 is the volume form on $\mathbb{P}^{n-1}$

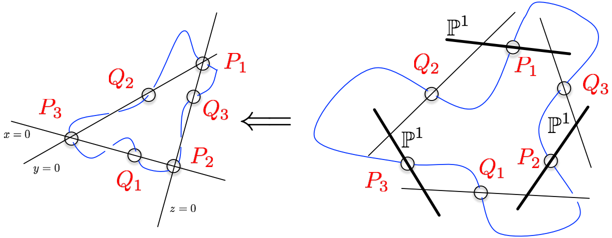
$$\Omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$$

$\omega_\Gamma(\underline{z}, \underline{x})$ is a function associated to a graph Γ such that the integrand is well defined in the completment of its polar locus in $\mathbb{P}^{n-1}$

The standard motto is Feynman integral *are* periods of the mixed Hodge structure after performing the appropriate blow-ups [Bloch, Esnault, Kreimer; Brown]

$$\mathfrak{M}_\Gamma := H^\bullet(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{Z_\Gamma}; \widetilde{\mathbb{\Delta}_n}\backslash\widetilde{\mathbb{\Delta}_n}\cap\widetilde{Z_\Gamma})$$

- Z_Γ is the singular locus of the integrand
- $\Delta_n \subset \mathbb{\Delta}_n := \{x_1 \cdots x_n = 0\}$ is in the normal crossings divisor
- Iterated blowups are needed to separated Z_Γ and $\mathbb{\Delta}_n$



The questions we want to answer are

- What kind of motives are appearing from Feynman integrals?
- Can we evaluate these periods integrals efficiently?

Standard type of Feynman integrals in QFT take a form

$$\omega_\Gamma(\underline{z},\underline{x}) := \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu-\frac{LD}{2}}} \prod_{i=1}^{n} x_i^{\nu_i-1}$$

with $\nu = \sum_{i=1}^{n} \nu_i$, $L \in \mathbb{N}$, $(D, \nu_1, \ldots, \nu_n) \in \mathbb{C}^{n+1}$.

An homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$ polynomial

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times \mathcal{L}(\underline{m}^2;\underline{x}) - \mathcal{V}_\Gamma(\underline{s},\underline{x})$$

- Homogeneous polynomial of degree L with $u_{a_1,\ldots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{a_1+\cdots+a_n=L} u_{a_1,\ldots,a_n} \prod_{i=1}^{n} x_i^{a_i}; \qquad a_i \in \{0,1\}$$

- the mass hyperplane $\mathcal{L}(\underline{m}^2;\underline{x}) := m_1^2 x_1 + \cdots + m_n^2 x_n$
- Homogeneous polynomial of degree $L_n + 1$

$$\mathcal{V}_\Gamma(\underline{x}) = \sum_{a_1+\cdots+a_n=L+1} s_{a_i,\ldots,a_n} \prod_{i=1}^{n} x_i^{a_i}; \qquad a_i \in \{0,1\}$$

the coefficients $s_{a_i,\ldots,a_n}$ are linear combination of the product of the external momenta $\underline{s} = \{p_i \cdot p_j\}$ and relation between these coefficients depend on the dimension D of space-time

Laurent polynomials

In $D = 2$ dimensions the sunset integrand is a Laurent polynomial $\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$

$$\Omega_\ominus^n = \frac{1}{\phi_\ominus^n(\underline{x})} \prod_{i=1}^n \frac{dx_i}{x_i}; \qquad \phi_\ominus^n(\underline{x}) = \left(\sum_{i=1}^n \frac{1}{x_i}\right)\left(\sum_{i=1}^n m_i^2 x_i\right) - p^2$$

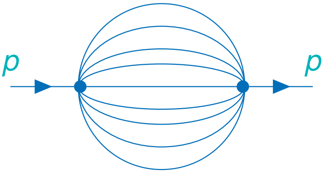
The classical period is obtained by integration over the torus $\mathbb{T}^n := \{|x_1| = \cdots = |x_n| = 1\}$ is the generalized Apéry series

$$\pi_\ominus = \int_{\mathbb{T}^n} \Omega_\ominus^n = \sum_{r_1+\cdots+r_n=n} \frac{1}{(p^2)^{n+1}} \left(\frac{n!}{r_1!\cdots r_n!}\right)^2 \prod_{i=1}^n (m_i^2)^{r_i}$$

We have pencil w.r.t p^2 with the singular fiber depending on the parameters in the Laurent polynomial
New efficient algorithms allow to compute the Picard-Fuchs equation with respect to p^2 [Lairez, Vanhove; de la Cruz, Vanhove]

The Sunset Feynman graph family

An important family of Feynman integral is the sunset graph



$$\Omega_n^{\ominus}(t,\underline{m}^2) := \frac{\Omega_0}{\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x})} \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_{\ominus})$$

$$\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x}) := x_1\cdots x_n\left(\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{j=1}^{n} m_j^2 x_j\right) - p^2\right)$$

- For $n=3$ we have periods of the elliptic curve

$$(x_1x_2+x_1x_3+x_2x_3)(m_1^2x_1+m_2^2x_2+m_3^2x_3) = p^2x_1x_2x_3$$

- For $n=4$ we have $K3$ of Pic from 19 to 16 depending on values of the parameters [Bloch, Kerr, V; Lairez, V;...]
- For $n=5$ this is a nodal CY 3-folds (containing the Hulek-Verrill as special case)

Sunset graphs toric variety $X_{p^2}(A_n)$ [Verrill]

The sunset graph polynomial

$$\mathcal{F}_n^{\ominus} = x_1 \cdots x_n \left(\left(\sum_{i=1}^{n} m_i^2 x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - p^2 \right)$$

is a character of the adjoint representation of A_{n-1} with support on the polytope generated by the A_{n-1} root lattice

- The Newton polytope Δ_n for $\mathcal{F}_n^{\ominus}$ is reflexive with only the origin as interior point
- The toric variety $X(A_{n-1})$ is the graph of the Cremona transformations $X_i \to 1/X_i$ of $\mathbb{P}^{n-1}$
  $X(A_{n-1})$ is obtained by blowing up the strict transform of the points, lines, planes etc. spanned by the subset of points $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$ in $\mathbb{P}^{n-1}$
- For the two-loop sunset integral we have the del Pezzo variety dP_6 (blowup of 3 points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$)

Fano Search

How many reflexive polytope are associated with Feynman integrals?

We can ask how many Fano variety can arise from Feynman integral?

When

$$\frac{L+1}{2}D-\nu=n_1,\qquad \nu-\frac{LD}{2}=n_2,\qquad (n_1,n_2)\in\mathbb{N}^2$$

we define a Laurent polynomial from

$$\Phi_{n_1,n_2}=\frac{\mathcal{U}^{n_1}(\mathcal{U}\mathcal{L}-\mathcal{V})^{n_2}}{x_1\cdots x_n}$$

homogeneous of degree 0 because $\deg(\mathcal{U}\mathcal{L}-\mathcal{V})\,n_2+\deg(\mathcal{U})\,n_1=n$
For an allowed (n_1,n_2) we have the two cases

- internal massless case $m_1=\cdots=m_n=0$, i.e. $\mathcal{L}=0$: $\frac{\mathcal{U}^{n_1}\mathcal{V}^{n_2}}{x_1\cdots x_n}$
- Vaccum graph $s_{\underline{a}}=0$, i.e. $\mathcal{V}=0$: $\frac{\mathcal{U}^{n_1+n_2}\mathcal{L}^{n_2}}{x_1\cdots x_n}$

One-loop graphs

One loop graphs have n edges and their Symanzik polynomials their graph polynomials read

$$\mathcal{U} = x_1 + \cdots + x_n; \qquad \mathcal{L} = m_1^2 x_1 + \cdots + m_n^2 x_n; \qquad \mathcal{V} = \sum_{1 \leqslant i < j \leqslant n} v_{ij} x_i x_j$$

Theorem (de la Cruz, Novichkov, V)

For all D such that $(D-n, n-D/2) \in \mathbb{N}^2$ the polytope is defined by as the convex hull of newton polygon of

$$\phi^{1-\mathrm{loop}} = \frac{\mathcal{U}^{D-n}(\mathcal{U}\mathcal{L} - \mathcal{V})^{n-D/2}}{x_1 \cdots x_n}$$

*is **reflexive** when $\mathcal{L} \neq 0$*
Reflexivity can be lost when $\mathcal{L} = 0$ when $D \neq 2n$

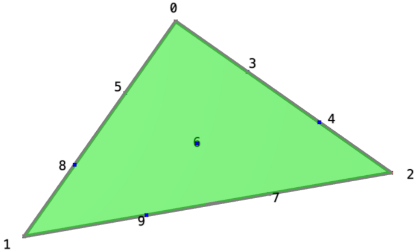
The triangle graph : massive

We consider the case $n = 3$ corresponding to the triangle graph

$$\mathcal{U}_3 = x_1 + x_2 + x_3, \mathcal{L}_3 = m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3$$

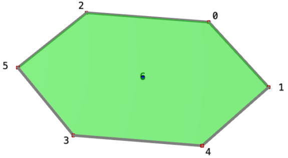
$$\mathcal{V}_3 = v_{12} x_1 x_2 + v_{13} x_1 x_3 + v_{23} x_2 x_3$$

Both $\phi_3^a = \frac{\mathcal{U}_3^3}{x_1 x_2 x_3}$ and $\phi_3^b = \frac{\mathcal{U}_3(\mathcal{U}_3 \mathcal{L}_3 - \mathcal{V}_3)}{x_1 x_2 x_3}$ both lead to the same reflexive polytopes



The triangle graph : massive

For $\mathcal{L}_3 = 0$ we have $\phi_3^b = (x_1 + x_2 + x_3)\left(\frac{v_{12}}{x_3} + \frac{v_{13}}{x_2} + \frac{v_{23}}{x_1}\right)$ which leads to a different reflexive polytope



$$\pi_3^b = \int_{\mathbb{T}^3} \frac{1}{(x_1 + x_2 + x_3)\left(\frac{v_{12}}{x_3} + \frac{v_{13}}{x_2} + \frac{v_{23}}{x_1}\right)} \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$$

is the $p^2 = 0$ value of the sunset integral

$$\pi_3^{\ominus}(p^2) = \int_{\mathbb{T}^3} \frac{1}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - p^2} \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$$

The pentagon graph

We consider $n = 5$ corresponding to the pentagon graph

$$\mathcal{U}_5 = x_1 + \cdots + x_5, \quad \mathcal{L}_5 = m_1^2 x_1 + \cdots + m_5^2 x_5, \quad \mathcal{V}_5 = \sum_{1 \leqslant i < j \leqslant 5} v_{ij} x_i x_j$$

We have reflexive lattice polytope

$$\phi^a_{(n,m)} = \frac{(\mathcal{U}_5 \mathcal{L}_5 - \mathcal{V}_5)^n \mathcal{U}_3^m}{x_1 \cdots x_5}, \quad (n, m) = (2, 1), (1, 3), (0, 5), \quad \mathcal{L}_5 \neq 0$$

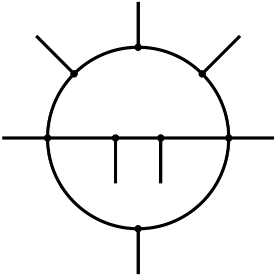
The lattice polytope is the same as one of the quintic in $\mathbb{P}^4$

$$x_1^5 + \cdots + x_5^5 - 5\psi x_1 \cdots x_5$$

But reflexivity is lost when $\mathcal{L}_5 = 0$ for

$$\psi^a_{(n,m)} = \frac{V_5^n U_3^m}{x_1 \cdots x_5} \quad (n, m) = (2, 1), (1, 3)$$

The two-loop graphs



For two-loop Feynman graphs of (a, b, c) vertices we consider the differential form

$$\Omega_{(a,b,c);D} = \frac{\mathcal{U}_{(a,b,c)}^{a+b+c-\frac{3D}{2}}}{(\mathcal{U}_{(a,b,c)}\mathcal{L}_{(a,b,c)} - \mathcal{V}_{(a,b,c);D})^{a+b+c-D}}$$

where $\deg \mathcal{U}_{(a,b,c)} = 2$ and $\deg \mathcal{F}_{(a,b,c);D} = 3$

$$\mathcal{U}_{(a,b,c)} = \left(\sum_{i=1}^{a} x_i\right)\left(\sum_{i=1}^{b} y_i\right) + \left(\sum_{i=1}^{a} x_i\right)\left(\sum_{i=1}^{c} z_i\right) + \left(\sum_{i=1}^{b} y_i\right)\left(\sum_{i=1}^{c} z_i\right)$$

$$\mathcal{L}_{(a,b,c)} = \sum_{i=1}^{a} m_{1i}^2 x_i + \sum_{i=1}^{b} m_{2i}^2 y_i + \sum_{i=1}^{c} m_{3i}^2 z_i$$

$$\mathcal{V}_{(a,b,c)} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} c(i,j,k) x_i y_j z_k$$

The two-loop graphs polytopes

We need to have the sum of the number of edges $a+b+c=3\mathbb{N}$, i.e. being in $\mathbb{P}^{3n-1}$

For $(a,b,c)=(1,1,1)$ this is the sunset and the toric variety dP_6

The next possible case is $a+b+c=6$, with $(a,b,c)=(2,2,2),(3,2,1),(4,1,1)$ where $\mathcal{V}_{(a,b,c)}$ is a cubic and $\mathcal{U}_{(a,b,c)}$ is degree 2 in $\mathbb{P}^5$

graph	$\mathcal{V}^2_{(a,b,c)}$	$(\mathcal{U}_{(a,b,c)}\mathcal{L}_{(a,b,c)}+\mathcal{V}_{(a,b,c)})^2$	$\mathcal{U}^3_{(a,b,c)}$
$(2,2,2)$	reflexive	not reflexive	reflexive
$(3,2,1)$	reflexive	not reflexive	not reflexive
$(4,1,1)$	not reflexive	not reflexive	not reflexive

We see that reflexivity become sparse and for $a+b+c=9$ there is no reflexive polytope.

After Kollár-Miyaoka-Mori we know that in any given dimension the number of Fano varieties is finite. The search for identifying them as Feynman integral is still on going with Leonardo de la Cruz and Pavel Novichkov.

Feynman Integrals are D-finite functions

Theorem [Kashiwara, Kawai; Petukhov, Smirnov; Bitoun et al.]
Feynman integrals are **holonomic D-finite functions** for generic values of $(D, \nu_1, \dots, \nu_n)$
For a given subset of the physical parameters $\underline{z} := (z_1, \dots, z_r) \subset \{\underline{s}, \underline{m}^2\}$ we want to derive a Gröbner basis of **minimal order** differential equations

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \partial_{\underline{z}}) \int_\sigma \Omega_\Gamma = \mathcal{S}_{\sigma,\Gamma}(\underline{z})$$

We construct differential operators $T_{\underline{z}}$ that annihilate the integrand in cohomology

$$T_{\underline{z}} \Omega_\Gamma = \mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial_{\underline{z}}}) \Omega_\Gamma + d\beta_\Gamma$$

- We ask that β_Γ is holomorphic on $\mathbb{P}^{n-1} \setminus Z_\Gamma$, i.e. it does not have poles that are not present in Ω_Γ

Feynman Integrals are twisted differential forms

In general dimension $D = 2\delta - 2\epsilon$ with $\delta \in \mathbb{N}$ and $\epsilon \in \mathbb{R}_+$ the Feynman integrals are twisted differential forms

$$I_\Gamma(\underline{s}, \underline{m}; \underline{\nu}, D) = \int_{\Delta_n} \Omega_\Gamma,$$

with

$$\Omega_\Gamma = \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu-\delta(L+1)}}{\mathcal{F}_\Gamma(\underline{x})^{\nu-\delta L}} \left(\frac{\mathcal{U}_\Gamma^{L+1}(\underline{x})}{\mathcal{F}_\Gamma^{L}(\underline{x})}\right)^{\epsilon} \prod_{i=1}^{n} x_i^{\nu_i-1}\, \Omega_0$$

The twists are well defined on projective space because they are powers of homogeneous degree 0 rational functions of $(x_1, \ldots, x_n)$.

For $\epsilon = 0$ and where choosen value of δ we get the Laurent polynomial previously discussed, but here we want to get to the general case

In $D=2\delta-2\epsilon$ dimensions with $\delta\in\mathbb{N}$ and $\epsilon\in\mathbb{C}$ we have a twisted differential form

$$\Omega_\Gamma=\frac{\mathcal{U}_\Gamma(\underline{x})^{\nu-\delta(L+1)}}{\mathcal{F}_\Gamma(\underline{x})^{\nu-\delta L}}\left(\frac{\mathcal{U}_\Gamma^{L+1}(\underline{x})}{\mathcal{F}_\Gamma^{L}(\underline{x})}\right)^{\epsilon}\prod_{i=1}^{n}x_i^{\nu_i-1}\,\Omega_0$$

We consider the partial derivative $\mathrm{a}=a_1+\cdots+a_r$

$$\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_\Gamma^\epsilon=\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{\mathrm{a}}}\,\Omega_\Gamma^\epsilon$$

- The locus $\mathcal{F}_\Gamma=0$ as non-isolated singularities. We need to use syzygies of $\mathrm{Jac}(\mathcal{F}_\Gamma)$
- For $\epsilon\neq0$ we have twisted differential form

We therefore adapt Griffith's pole reduction for overcome these difficulties.

Extended Griffiths' pole reduction

Reducing $P^{(a_1,\dots,a_r)}(\underline{x})$ in the Jacobian ideal of $\mathcal{F}_\Gamma$

$$P^{(a_1,\dots,a_r)}(\underline{x}) = \vec{C}_{\mathrm{a}}(\underline{x}) \cdot \vec{\nabla}\mathcal{F}_\Gamma\,,$$

We introduce the differential twisted form

$$\beta^{(a_1,\dots,a_r)} = \sum_{1\leqslant i<j\leqslant n} \frac{x_i C^j_{\mathrm{a}}(\underline{x}) - x_j C^i_{\mathrm{a}}(\underline{x})}{\mathcal{F}_\Gamma^{\mathrm{a}-1}}\,\Omega^\epsilon_\Gamma\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n$$

Following Griffiths' pole reduction we reduce the pole order of $\mathcal{F}_\Gamma$

$$\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{a}}\Omega_\Gamma = \frac{\vec{\nabla}\cdot\vec{C}_{\mathrm{a}}(\underline{x}) + \lambda_U \vec{C}_{\mathrm{a}}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma}{(\mathrm{a}-1+\lambda_F)\,\mathcal{F}_\Gamma^{\mathrm{a}-1}}\Omega^\epsilon_\Gamma + \frac{d\beta_\Gamma^{(a_1,\dots,a_r)}}{\mathrm{a}-1+\lambda_F}$$

where we have defined

$$\lambda_U = n-(L+1)(\delta-\epsilon), \qquad \lambda_F = n-L(\delta-\epsilon).$$

Extended Griffiths' pole reduction

The term $\vec{C}_{\mathrm{a}} \cdot \vec{\nabla} \log \mathcal{U}_\Gamma$ which has a pole in $\mathcal{U}_\Gamma$, which is reduced by asking that

$$\vec{C}_{\mathrm{a}}(\underline{x}) \cdot \vec{\nabla}\mathcal{U}_\Gamma = c_{\mathrm{a}}(\underline{x})\,\mathcal{U}_\Gamma\,,$$

which is equivalent to the computation of syzygies of $\mathrm{Jac}(\mathcal{U}_\Gamma)$ using the homogeneity of $\mathcal{U}_\Gamma$

$$\left(L\vec{C}_{\mathrm{a}}(\underline{x}) - c_{\mathrm{a}}(\underline{x})\vec{x}\right) \cdot \vec{\nabla}\mathcal{U}_\Gamma = 0\,,$$

Solving the linear system

$$\begin{cases} \vec{C}_{\mathrm{a}}(\underline{x}) \cdot \vec{\nabla}\mathcal{F}_\Gamma = P^{(a_1,\dots,a_r)}(\underline{x}) \\ \vec{C}_{\mathrm{a}}(\underline{x}) \cdot \vec{\nabla}\mathcal{U}_\Gamma = c_{\mathrm{a}}(\underline{x})\mathcal{U}_\Gamma \end{cases},$$

we have the pole reduction

$$\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{a}}\Omega_\Gamma = \frac{\vec{\nabla}\cdot\vec{C}_{\mathrm{a}}(\underline{x}) + \lambda_U\, c_{\mathrm{a}}(\underline{x})}{(\mathrm{a}-1+\lambda_F)\mathcal{F}_\Gamma^{\mathrm{a}-1}}\Omega_\Gamma^{\epsilon} + \frac{\mathrm{a}-1}{\mathrm{a}+n-L(\delta-\epsilon)}d\beta_\Gamma^{(a_1,\dots,a_r)}$$

Extended Griffiths-Dwork: syzygies

For singular hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$ the Jacobian reduction may not be enough to reduce the pole order when $k \geqslant n$

For instance non-isolated singularities arise for the sunset family from $n \geqslant 4$ (i.e. $K3$ and higher CY), as Jacbian vanishes for $n \geqslant 4$ for

$$x_i = x_j = 0, \qquad m_1^2 x_1 + \cdots + m_n^2 x_n^2 = 0 \qquad i \neq j$$

Therefore we need to take into account the syzygies because of that.

Instead of computing the resolution separatly we do solve the linear system from the reduction in the Jacobian of $\mathcal{F}$ and $\mathcal{U}$.

This leads to a large linear systems that are solved using finite field method implemented in the code `FiniteFlow` by [Peraro].

Differential operators I

We turn to the construction of the differential operator

$$\mathcal{L}_\Gamma^\epsilon = \sum_{a=0}^{N(\Gamma,\epsilon)} \sum_{\substack{a=a_1+\cdots+a_r\\ a_i\geqslant 0}} c_{a_1,\dots,a_r}(\vec{m},\vec{s},\epsilon,\kappa)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}$$

such that

$$\mathcal{L}_\Gamma^\epsilon \Omega_\Gamma^\epsilon = d\beta_\Gamma^\epsilon .$$

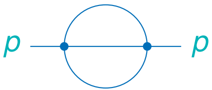
We start with the pole reduction with taking a first order derivative, and increase the order until the extended Griffith's pole reduction closes.

Holonomicity of Feynman integrals imply that the algorithm finishes because the order $N(\Gamma,\epsilon)$ has for upper bound

$$N(\Gamma,\epsilon) \leqslant \dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n\backslash \mathbb{V}(\mathcal{U}_\Gamma)\cup\mathbb{V}(\mathcal{F}_\Gamma)\right)$$

As it turns for many cases we have a strict inequality

The sunset graph in $D = 2 - 2\epsilon$ dimensions



$$I^{\epsilon}_{\ominus}(p^2, \underline{m}^2) = \int_{\mathbb{R}^3_+} \left(\frac{\mathcal{U}^3_{\ominus}}{\mathcal{F}^2_{\ominus}} \right)^{\epsilon} \frac{dx_1\, dx_2 dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

$$\mathcal{F}_{\ominus}(\underline{x}) = (x_1 x_2 + x_1 x_3 + x_2 x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - p^2 x_1 x_2 x_3$$

- $\mathcal{F}_{\ominus}(\underline{x}) = 0$ defines an elliptic curve $\mathcal{E}_{\ominus}$
- For $\epsilon = 0$ the integral is an elliptic dilogarithm in the regulator of a class in the motivic cohomology of the elliptic curve $\mathcal{E}_{\ominus}$ [Bloch, Vanhove; Bloch, Kerr, Vanhove]

The two-loop sunset graph in general dimensions

$$I_{\ominus}^{\epsilon}(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \left(\frac{\mathcal{U}_{\ominus}^3}{\mathcal{F}_{\ominus}^2}\right)^{\epsilon} \frac{dx_1\, dx_2\, dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

Applying the algorithms we find a fourth order differential equation

$$\mathcal{L}_{\ominus}^{\epsilon} = \mathcal{L}_1^{(1)} \mathcal{L}_1^{(2)} \mathcal{L}_2^{\ominus} + \epsilon \mathcal{L}_4^{(3)} + \epsilon^2 \mathcal{L}_3^{(4)} + \epsilon^3 \mathcal{L}_2^{(5)} + \epsilon^4 \mathcal{L}_1^{(6)} + \epsilon^5 \mathcal{L}_0^{(7)},$$

- The differential equation is irreducible for generic ϵ which we checked using the algorithms of [Chyzak, Goyer, Mezzarobba]
- The operators $\mathcal{L}_r^{(a)}$ are of order r
- The $\epsilon = 0$ term factorizes $\mathcal{L}_1^{(1)} \mathcal{L}_1^{(2)} \mathcal{L}_2^{\ominus}$

The sunset graph: Gauß-Manin connexion

If one considers a family of elliptic curve E

$$y^2 = 4x^3 - g_2(t)x - g_3(t); \quad j(t) = \frac{g_2(t)^3}{\Delta(t)}; \quad \delta(t) = 3g_3(t)\frac{d}{dt}g_2(t) - 2g_2(t)\frac{d}{dt}g_3(t)$$

the periods satisfy the differential system of equations

$$\frac{d}{dt}\begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d}{dt}\log\Delta(t) & \frac{3\delta(t)}{2\Delta(t)} \\ -\frac{g_2(t)\delta(t)}{8\Delta(t)} & \frac{1}{12}\frac{d}{dt}\log\Delta(t) \end{pmatrix} \begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix}$$

The Picard–Fuchs operator acting on the period integral $\int_\gamma dx/y$ is

$$\mathcal{L}_2^E = 144\Delta(t)^2\delta(t)\frac{d^2}{dt^2} + 144\Delta(t)\left(\delta(t)\frac{d\Delta(t)}{dt} - \Delta(t)\frac{d\delta(t)}{dt}\right)\frac{d}{dt}$$
$$+ 27g_2(t)\delta(t)^3 + 12\frac{d^2\Delta(t)}{dt^2}\delta(t)\,\Delta(t) - \left(\frac{d\Delta(t)}{dt}\right)^2\delta(t) - 12\frac{d\delta(t)}{dt}\Delta(t)\,\frac{d\Delta(t)}{dt}.$$

The two-loop sunset graph in general dimensions

$$I^{\epsilon}_{\ominus}(p^2, \underline{m}^2) = \int_{\mathbb{R}^3_+} \left(\frac{\mathcal{U}^3_{\ominus}}{\mathcal{F}^2_{\ominus}}\right)^{\epsilon} \frac{dx_1 dx_2 dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

$$\mathcal{L}^{\epsilon}_{\ominus} = \mathcal{L}^{(1)}_1 \mathcal{L}^{(2)}_1 \mathcal{L}^{\ominus}_2 + \epsilon \mathcal{L}^{(3)}_4 + \epsilon^2 \mathcal{L}^{(4)}_3 + \epsilon^3 \mathcal{L}^{(5)}_2 + \epsilon^4 \mathcal{L}^{(6)}_1 + \epsilon^5 \mathcal{L}^{(7)}_0 ,$$

- The differential operator $\mathcal{L}^{\ominus}_2$ is the Picard-Fuchs operator for the elliptic curve defined by
- The deformation ϵ affects only the apparent singularities

$$\mathcal{L}^{\epsilon}_{\ominus}\Big|_{\left(\frac{d}{dt}\right)^4} = \Delta(t)^2 \Big(-(2\epsilon + 5)\, t^2 - 2\left(m_1^2 + m_2^2 + m_3^2\right)(1 + 2\epsilon)\, t + (7 + 6\epsilon) \prod_{i=1}^{4} \mu_i \Big)$$

where $\Delta(t) = t^3 \prod_{i=1}^4 (t - \mu_i^2)$ is the discriminant of the elliptic curve $\mathcal{F}_{\ominus} = 0$

Apparent singularities

For a differential equation

$$c_N(z)\frac{d^N f(z)}{dz^N} + \cdots + c_0(z)f(z) = 0,$$

the roots of $c_N(z)$ are the singularities of the differential equation. A root of $c_N(z)$ where the solution $f(z)$ is regular is called an apparent singularity. A root of $c_N(z)$ where the solution has a singularity is a real singularity

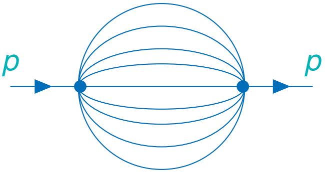
For the case of Feynman integrals the non-apparent (real) singularities are the roots of the discriminant of the singular locus of the integrand of Feynman integrals

Theorem (de la Cruz, V)

The parameter ϵ appears only in the apparent singularities of the differential operator $\mathcal{L}_z^\epsilon$. This means that the ϵ deformation does not change the position of the real singularities, but it affects the local behaviour (the monodromy) of the solution near the singularity.

Sunset graph Picard–Fuchs operator

Applying the algorithm to the higher dimensional family of sunset graph



$$\Omega_n^{\ominus}(t, \underline{m}^2) := \frac{\Omega_0}{\mathcal{F}_n^{\ominus}(t, \underline{m}^2; \underline{x})} \in H^{n-1}(\mathbb{P}^{n-1} \backslash X_{\ominus})$$

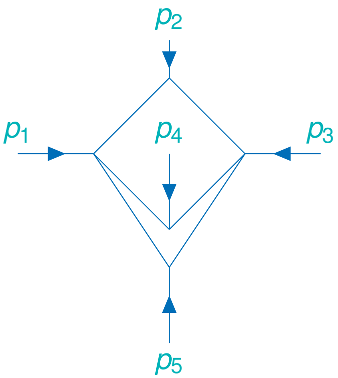
$$\mathcal{F}_n^{\ominus}(t, \underline{m}^2; \underline{x}) := x_1 \cdots x_n \left(\left(\sum_{i=1}^{n} \frac{1}{x_i} \right) \left(\sum_{j=1}^{n} m_j^2 x_j \right) - t \right)$$

For generic physical parameters we find a minimal order Picard–Fuchs operator

$$\mathcal{L}_t = \sum_{r=0}^{o_n} q_r(t, \underline{m}^2) \left(\frac{d}{dt} \right)^r \qquad o_n = 2^n - \binom{n+1}{\left\lfloor \frac{n+1}{2} \right\rfloor}; n \geqslant 2.$$

supporting that we have relative periods of a Calabi–Yau of dimension $n-2$. Which is in total agreement with Verrill toric analysis.

Tardigrade



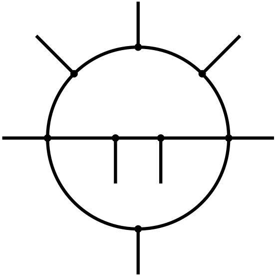
The rational differential form in $\mathbb{P}^5$

$$\Omega_{(2,2,2)}(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_{(2,2,2)}\mathcal{L}_{(2,2,2)} - t\mathcal{V}_{(2,2,2)}\right)^2}$$

$$\mathcal{U}_6(\underline{x}) = (x_1+x_2)(x_3+x_4) + (x_1+x_2)(x_5+x_6) + (x_3+x_4)(x_5+x_6)$$

$\mathcal{V}_6(\underline{s},\underline{x}) = \sum_{1\leqslant i,j,k\leqslant 6} C_{ijk} y_i y_j y_k$ with linear changes $(x_{2i-1}, x_{2i}) \to (y_{2i-1}, y_{2i})$ and $i = 1,2,3$ C_{ijk} symmetric traceless i.e. $C_{iij} = 0$

- the algorithm gives an irreducible Picard–Fuchs operator of order 11 with an head polynomial of degree up to 215
- Using [Eric Pichon-Pharabod] program, we have confirmed that this is K3 an elliptically fiber K3 with singular fibres $14I_1 \oplus 2I_4 \oplus I_2$ of Picard number 11



For two-loop Feynman graphs of (a, b, c) vertices we consider the differential form

$$\Omega_{(a,b,c);D} = \frac{\mathcal{U}_{(a,b,c)}^{a+b+c-\frac{3D}{2}}}{\mathcal{F}_{(a,b,c);D}^{a+b+c-D}}\Omega_0$$

where $\deg \mathcal{U}_{(a,b,c)} = 2$ and $\deg \mathcal{F}_{(a,b,c);D} = 3$

We determine the mixed Hodge structure

$$\mathcal{H}^{a+b+c-1} := H^{a+b+c-1}(\mathbb{P}^{\widetilde{a+b+c}-1}\backslash \widetilde{Z}_{(a,b,c)}; \widetilde{\Pi}_n \backslash \widetilde{\Pi}_n \cap \widetilde{Z}_{(a,b,c)})$$

The method is based on quadric fibrations. The cohomology of $Z_{(a,b,c)}$ is obtained by iterated extensions with Tate twists of cohomology of hyperelliptic curves and Tate Hodge structure

We have presented some examples $(a, b, c) = (1, 1, 1)$ sunset, and $(a, b, c) = (2, 2, 2)$ tardigrade

Definitions

Definition

1. Let $\mathbf{MHS}_{\mathbb{Q}}$ denote the abelian category of $\mathbb{Q}$-mixed Hodge structures.
2. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(C;\mathbb{Q})$ for every hyperelliptic curve C is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.
3. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(E;\mathbb{Q})$ for every elliptic curve E is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}$.

We have the following results for the Hodge structure of two-loop planar graphs

C. F. Doran, A. Harder, E. Pichon-Pharabod and P. Vanhove,
Motivic geometry of two-loop Feynman integrals
[arXiv:2302.14840]

Graph $(a, 1, c)$

Theorem

1. *If* $3D/2 \leqslant a + c$ *then* $\mathcal{H}^{a+c} \in \mathbf{MHS}^{\mathrm{hyp}}_{\mathbb{Q}}$.
2. *Suppose* $a \leqslant 2$ *or* $c \leqslant 2$. *Then* $\mathcal{H}^{a+c-1}(Z_{(a,1,c)}; \mathbb{Q}) \in \mathbf{MHS}^{\mathrm{ell}}_{\mathbb{Q}}$.

Corollary

If $3D/2 \leqslant a + c$ *and either* $a \leqslant 2$ *or* $c \leqslant 2$ *then* $\mathcal{H}^{a+c} \in \mathbf{MHS}^{\mathrm{ell}}_{\mathbb{Q}}$.

This means that the mixed Hodge structure $\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - \widetilde{Z}_{(a,1,c)}; B - (B \cap \widetilde{Z}_{(a,1,c)}))$ is constructed by taking iterated extensions of $\mathrm{H}^1(E; \mathbb{Q})(-a)^{r_1}$ and $\mathbb{Q}(-b)^{r_2}$ for different values of a, b, r_1, and r_2, and with various possibly different elliptic curves.
Therefore the Feynman integrals in this cases are built from algebraic and elliptic functions.

Graph $(3,1,3)$ double-box

$$\Omega_{(3,1,3);D}(t) = \frac{\mathcal{U}_{(3,1,3)}(\underline{x})\Omega_0}{(\mathcal{U}_{(3,1,3)}(\underline{x})\mathcal{L}_{(3,1,3)}(\underline{m}^2,\underline{x}) - t\mathcal{V}_{(3,1,3);D}(\underline{s},\underline{x}))^3}$$

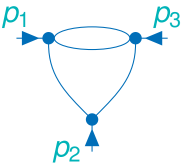
Theorem

For arbitrary kinematic parameters, and arbitrary space-time dimension D, $W_4\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q})$ is mixed Tate.

1. *If $D \geqslant 5$ then $\mathrm{Gr}_5^W\mathrm{H}^5(Z_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(C;\mathbb{Q})(-2)$ for a curve C which has genus 2 for generic kinematic parameters.*
2. *If $D = 4$ then $\mathrm{Gr}_5^W\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q}) \cong \mathrm{H}^1(E;\mathbb{Q})(-2)$ for a curve E which is elliptic for generic kinematic parameters.*
3. *If $D < 4$ then $\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q})$ is mixed Tate.*

In $D = 4$ the PF operator obtained by the extended Griffith–Dwork construction is *identical* to the one associated with the canonical differential form on the elliptic curve defined from the graph polynomial

Graph $(1,1,2)$ ice-cream



With a regular β the Picard–Fuchs operator is of order 2 and degree 9

$$\mathcal{L}_t^2 = q_0(t) + q_1(t)\frac{d}{dt} + q_2(t)\left(\frac{d}{dt}\right)^2,$$

$\mathcal{L}_t$ is a Liouvillian differential equations with only rational solutions

Proposition

- $\mathrm{H}^2(Z_{(2,1,1)}(t);\mathbb{Q})$ *is generically pure Tate.*
- *Let* $Z = V(Disc(Z_{(2,1,1)})(x,z,t)) \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ *and we may define* $\mathbb{V} := \pi_* \underline{\mathbb{Q}}_Z$*. The local system* $\mathbb{V}$ *is isomorphic to the direct sum* $\mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \underline{\mathbb{Q}}_B^2$ *where* $\mathbb{U}_1$ *is a rank 1 local system and* $\mathbb{U}_2$ *is a rank 2 local system*
- *For generic kinematic and mass parameters,* $Sol(\mathcal{L}_{(2,1,1)})$ *is isomorphic to* $\mathbb{U}_2 \cong \mathbb{U}_2^\vee$*.*

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Lemma

Let $\mathbb{L}_1, \mathbb{L}_2$ be local systems of rank 1, and suppose that $\mathbf{s}_1$ and $\mathbf{s}_2$ are sections of $\mathbb{L}_1 \otimes \mathcal{O}_M$ and $\mathbb{L}_2 \otimes \mathcal{O}_M$ respectively. If

$$\mathcal{L}_{\mathbf{s}_1} = \frac{d}{ds} - f_1(s), \qquad \mathcal{L}_{\mathbf{s}_2} = \frac{d}{ds} - f_2(s)$$

are the differential equations associated to $\mathbf{s}_1$ and $\mathbf{s}_2$ respectively, then the differential equation associated to the section $\mathbf{s}_1 \oplus \mathbf{s}_2$ of $(\mathbb{L}_1 \oplus \mathbb{L}_2) \otimes \mathcal{O}_M$ is

$$\mathcal{L}_{\mathbf{s}_1 \oplus \mathbf{s}_2} = (f_1(s) - f_2(s)) \frac{d^2}{ds^2} + (f_2(s)^2 - f_1(s)^2 - f_1'(s) + f_2'(s)) \frac{d}{ds} + f_2(s) f_1'(s) - f_1(s) f_2'(s) + f_1(s)^2 f_2(s) - f_1(s) f_2(s)^2$$

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Proposition

After the base change

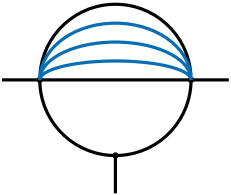
$$t = \frac{(m_1 - m_2)^2 s^2 + (m_1 + m_2)^2}{p_2^2 (s^2 + 1)},$$

the differential operator $\mathcal{L}_{(2,1,1);2}$ *is of the form given previously where* $f_i(s)$ *with* $i = 1, 2$ *are obtained from the application of the change of variables* $T = \rho_i(s)$ *with* $i = 1, 2$ *to the differential operator*

$$\frac{d}{dT} - \frac{(m_3^2 + m_4^2 - T)}{((m_3 - m_4)^2 - T)((m_3 + m_4)^2 - T)} \Longrightarrow \frac{d}{ds} - f_i(s)$$

with $T = \rho_i(s)$ *and* $i = 1, 2$ *are roots of discriminant obtained from blowing up the linear subspace* $x_1 = z = 0$

multiscoop ice-cream



For the multiscoop ice cream cone families there is a conic fibration on $X_{(2,[1]^k);D}$, and the discriminant locus this conic fibration is a union of two Calabi–Yau $(k-2)$-folds associated to the $(k-1)$-loop sunset graph

Therefore $\mathrm{Gr}_k^W \mathrm{H}^k(Z_{(2,[1]^k)};\mathbb{Q})$ arises from

$$\mathrm{H}^{k-2}(Z^{(1)}_{([1]^k)};\mathbb{Q}) \oplus \mathrm{H}^{k-2}(X^{(2)}_{([1]^k)};\mathbb{Q})$$

where $Z^{(1)}_{([1]^k)}$ and $Z^{(2)}_{([1]^k)}$ are distinct $(k-1)$-loop sunset Calabi–Yau $(k-2)$-folds. This is supported by the computations of the PF operator for the 2-scoop ice-cream which is of rank 4. In this case $Z^{(1)}_{(1,1,1)}$ and $Z^{(2)}_{(1,1,1)}$ are elliptic curves, so the rank of $\mathcal{L}_{(2,1,1,1)}$ agrees with the rank of $\mathrm{H}^1(Z^{(1)}_{(1,1,1)};\mathbb{Q}) \oplus \mathrm{H}^1(Z^{(2)}_{(1,1,1)};\mathbb{Q})$.

Conclusion

We have presented result toward a classification of the motives (mixed Hodge structures) associated to Feynman graphs

- We have started a classification of the reflexive polytope from Feynman integral. At one-loop reflexivity allows exists, but from two-loop this is becoming more sparse.
- Can we achieve a complete classification?
- We have remarked that different graph share the same polytope, which can be explained by specializing parameters. What about geometric transitions?

We have presented that an extension of the Griffiths-Dwork algorithm for computing differential operators for Feynman integrals

- The algorithm work for non-smooth case (which is the generic case for Feynman integral) and beyond the rational case
- The derived Picard-Fuchs operators are compatible with the explict Hodge theoretical construction when available.