Maximal connected algebraic subgroups of the real Cremona group (j.w. with Susanna Zimmermann)

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References:

- *Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces*, with Jérémy Blanc and Andrea Fanelli. Épijournal de Géométrie Algébrique, Volume 6, January 2023 (47 pages).
- *Connected algebraic groups acting on 3-dimensional Mori fibrations*, with Jérémy Blanc and Andrea Fanelli. International Mathematics Research Notices, Volume 2023, Issue 2, January 2023, Pages 1572-1689.
- *Real forms of Mori fiber spaces with many symmetries*, with Susanna Zimmermann. Preprint arXiv:2403.14493.

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1. *For a given complex variety X, does there exist a real form of X?*
2. *If yes, how many (up to isomorphism)?*

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- $\mathrm{k} = \overline{\mathrm{k}}$ has characteristic zero and $n = 3$ (Enriques-Fano 1898, Umemura 1980's, Blanc-Fanelli-T. 2021-2023).

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- *the Weil restriction* $\mathcal{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$ *(which is a rational real form of* $\mathbb{F}_{0,\mathbb{C}}$*).*

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(a)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{F}_a^{b,c} \longrightarrow \mathbb{F}_a$	*with* $a, b \geq 0$, $a \neq 1$, $c \in \mathbb{Z}$, *and* $(a,b,c) = (0,1,-1)$; *or* $a = 0$, $c \neq 1$, $b \geq 2$, $b \geq \lvert c \rvert$; *or* $-a < c < a(b-1)$; *or* $b = c = 0$.
(b)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{P}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 2$.
(c)	*An Umemura*	$\mathbb{P}^1$*-bundle*	$\mathcal{U}_a^{b,c} \longrightarrow \mathbb{F}_a$	*for some* $a, b \geq 1, c \geq 2$ *with* $c < b$ *if* $a = 1$; *and* $c - 2 < ab$ *and* $c - 2 \neq a(b-1)$ *if* $a \geq 2$.
(d)	*A Schwarzenberger*	$\mathbb{P}^1$*-bundle*	$\mathcal{S}_b \longrightarrow \mathbb{P}^2$	*for some* $b = 1$ *or* $b \geq 3$.
(e)	*A*	$\mathbb{P}^1$*-bundle*	$\mathcal{V}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 3$.
(f)	*A singular*	$\mathbb{P}^1$*-fibration*	$\mathcal{W}_b \longrightarrow \mathbb{P}(1,1,2)$	*for some* $b \geq 2$.
(g)	*A decomposable*	$\mathbb{P}^2$*-bundle*	$\mathcal{R}_{m,n} \longrightarrow \mathbb{P}^1$	*for some* $m \geq n \geq 0$, *with* $(m,n) \neq (1,0)$ *and* $m = n$ *or* $m > 2n$.
(h)	*An Umemura*	*quadric fibration*	$\mathcal{Q}_g \longrightarrow \mathbb{P}^1$	*for some homogeneous polynomial* $g \in k[u_0, u_1]$ *of even degree with at least four roots of odd multiplicity.*

(i) *A rational $\mathbb{Q}$-factorial Fano threefold of Picard rank 1 with terminal singularities.*

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Partial conclusion: The connected algebraic subgroups of Cr_n are those that act (regularly) on rational Mori fiber spaces.

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If X is a real rational projective variety such that $\mathrm{Aut}^\circ(X_\mathbb{C})$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3_\mathbb{C})$, then $\mathrm{Aut}^\circ(X)$ is a maximal connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3_\mathbb{R})$.

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Theorem ($\mathrm{k} = \overline{\mathrm{k}}$, $n = 3$, Theorem E in [BFT21])

If G is a connected algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^3_k)$, then G is conjugate to an algebraic subgroup of $\mathrm{Aut}^\circ(X)$, where $X \to Y$ is one of the following Mori fiber spaces :

(a)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{F}_a^{b,c} \longrightarrow \mathbb{F}_a$	*with* $a, b \geq 0$, $a \neq 1$, $c \in \mathbb{Z}$, *and* $(a,b,c) = (0,1,-1)$; *or* $a = 0$, $c \neq 1$, $b \geq 2$, $b \geq \lvert c \rvert$; *or* $-a < c < a(b-1)$; *or* $b = c = 0$.
(b)	*A decomposable*	$\mathbb{P}^1$*-bundle*	$\mathcal{P}_b \longrightarrow \mathbb{P}^2$	*for some* $b \geq 2$.
(c)	*An Umemura*	$\mathbb{P}^1$*-bundle*	$\mathcal{U}_a^{b,c} \longrightarrow \mathbb{F}_a$	*for some* $a, b \geq 1, c \geq 2$ *with* $c < b$ *if* $a = 1$; *and* $c - 2 < ab$ *and* $c - 2 \neq a(b-1)$ *if* $a \geq 2$.
(d)	*A Schwarzenberger*	$\mathbb{P}^1$*-bundle*	$\mathcal{S}_b \longrightarrow \mathbb{P}^2$	*for some* $b = 1$ *or* $b \geq 3$.
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(f)	*A singular*	$\mathbb{P}^1$*-fibration*	$\mathcal{W}_b \longrightarrow \mathbb{P}(1,1,2)$	*for some* $b \geq 2$.
(g)	*A decomposable*	$\mathbb{P}^2$*-bundle*	$\mathcal{R}_{m,n} \longrightarrow \mathbb{P}^1$	*for some* $m \geq n \geq 0$, *with* $(m,n) \neq (1,0)$ *and* $m = n$ *or* $m > 2n$.
(h)	*An Umemura*	*quadric fibration*	$\mathcal{Q}_g \longrightarrow \mathbb{P}^1$	*for some homogeneous polynomial* $g \in k[u_0, u_1]$ *of even degree with at least four roots of odd multiplicity.*

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Subgroup $F \subseteq \mathrm{PGL}_2(\mathbb{C})$	n even $(n \geq 2)$			n odd $(n \geq 3)$		
	rational	?	w/o real points	rational	?	w/o real points
$A_l,\ l \geq 1,\ l$ odd	2	2	0	2	2	0
$A_l,\ l \geq 2,\ l$ even	4	4	0	2	2	0
$D_l,\ l \geq 3,\ l$ odd	4	4	0	2	2	0
$D_l,\ l \geq 2,\ l$ even	6	6	4	3	3	4
E_6	2	2	4	1	1	4
E_7	4	4	4	2	2	4
E_8	2	2	4	1	1	4

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- The automorphism group of $\mathrm{Aut}(\mathcal{Q}_g)$ admits the following sequence:

$$1 \to \mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \to \mathrm{Aut}(\mathcal{Q}_g) \to F \to 1,$$

  where F is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ and $\mathrm{Aut}(\mathcal{Q}_g)_{\mathbb{P}^1} \simeq \mathrm{PGL}_2(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$.
- The number of real forms of $\mathcal{Q}_{g,\mathbb{C}}$, which depends on F, is shown in the following table:

Subgroup $F \subseteq \mathrm{PGL}_2(\mathbb{C})$	n even $(n \geq 2)$			n odd $(n \geq 3)$		
	rational	?	w/o real points	rational	?	w/o real points
$A_l,\ l \geq 1,\ l$ odd	2	2	0	2	2	0
$A_l,\ l \geq 2,\ l$ even	4	4	0	2	2	0
$D_l,\ l \geq 3,\ l$ odd	4	4	0	2	2	0
$D_l,\ l \geq 2,\ l$ even	6	6	4	3	3	4
E_6	2	2	4	1	1	4
E_7	4	4	4	2	2	4
E_8	2	2	4	1	1	4

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- What are the maximal connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$ when $n \geq 4$? Is there a pattern?

Thank you for your attention!