From Catalan numbers to integrable dynamics:

Continued fractions & Hankel determinants

for q-numbers

(joint work /w Emmanuel Pedon)

0. **Goal:**

$C_n = 1, 1, 2, 5, 14, 42, \dots$ $M_n = 1, 1, 2, 4, 9, 21, 51, \dots$

Catalan Motzkin

Integrable dynamical systems

"quantized numbers"

1.) Hankel determinants

$\left(a_n\right)_{n\in\mathbb{N}}$ (integer) sequence $\rightsquigarrow$ Δ_n, $\Delta_n^{(1)}$, $\Delta_n^{(11)}$

Δ_n: $\Delta_0 := 1$, $\Delta_1 = a_1$, $\Delta_2 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \dots$

$\Delta_n^{(k)}$: $\Delta_0^{(k)} := 1$, $\Delta_1^{(k)} = a_{k+1}$, $\Delta_2^{(k)} = \begin{vmatrix} a_{k+1} & a_{k+2} \\ a_{k+2} & a_{k+3} \end{vmatrix} \dots$

Classical theorems:

Thm 1. (I) $\Delta_n(C) = \Delta_n^{(1)}(C) = 1, 1, 1, 1 \ldots$

(II) $\Delta_n(M) = 1, 1, 1, 1, \ldots$

(Aigner ~1998) $\Delta'_n(M) = 1, 1, 0, -1, -1, 0, \ldots$

3-antiperiodic

C_n, M_n are characterised by this.

2. What is... a q-analogue?

q-analogues: $\left\{ \begin{array}{l} \text{quantum groups} \\ \text{knot invariants} \\ \text{combinatorics} \\ \ldots \\ \text{quantization} \end{array} \right.$

q-analogues of real numbers

(Sophie Morier-Genoud – V. O. ≈ 2020).

q-integers: $\quad n \in \mathbb{N}$

(Euler ≈ 1760) $\quad [n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}$

q-binomials: $\quad \binom{n}{m}_q = \dfrac{[n]_q!}{[n-m]_q!\,[m]_q!}$

(Gauss ≈ 1808)

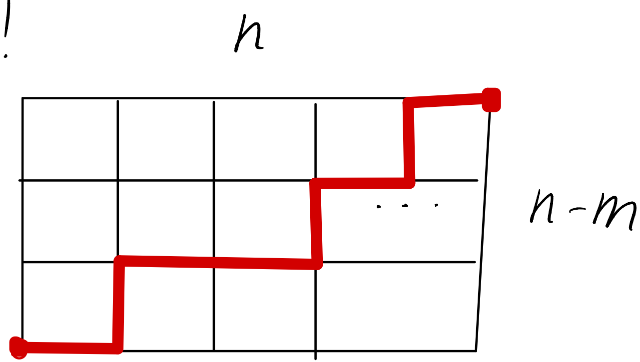
where $[n]_q! = [1]_q [2]_q [3]_q \cdots [n]_q$

Polynomials in q $\quad$ Ex. $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$

$$\binom{n}{m}_q = \binom{n-1}{m-1}_q + q^m \binom{n-1}{m}_q$$

Counting Better!

$\binom{n}{m}$



\# NE path

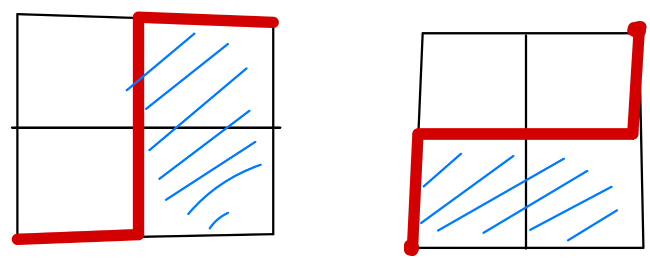
Thm

$$\binom{n}{m}_q = \sum_{k \geq 1} P_k \, q^k$$

P_k = # NE path with k boxes under path

Ex.

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$



Other properties

Thm $\binom{n}{m}_q = \#$ points in $Gr_{m,n}(\mathbb{F}_q)$

Thm $(x+y)^n = \sum_{m=0}^{n} \binom{n}{m}_q x^m y^{n-m}$ $\boxed{yx = qxy}$

Thm Sylvester (≈ 1878)

Polynomials $\binom{n}{m}_q$ are unimodal

$$= 1 + P_1 q + P_2 q^2 + \cdots$$

3.] q-rationals $\frac{n}{m} \in \mathbb{Q}$, what is $\left[\frac{n}{m}\right]_q$?

Geometric idea: group of symmetry!

$PSL(2,\mathbb{Z})$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$, $a,b,c,d \in \mathbb{Z}$

acts on $\mathbb{Q} \cup \{\infty\}$, $\infty = \frac{1}{0}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \dfrac{ax+b}{cx+d}$

Want a map $[\ ]_q : \mathbb{Q} \cup \{\infty\} \to \mathbb{Z}(q)$

commuting w $PSL(2,\mathbb{Z})$-action

rational func-s in q

$PSL(2,\mathbb{Z})$ - action on $\mathbb{Z}(q)$:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ - generators}$$

$$T \cdot x = x + 1 \qquad S \cdot x = -\frac{1}{x}$$

<u>Def.</u> $\quad T_q \cdot X = qX + 1, \quad S_q \cdot X = -\frac{1}{qX}$

where $X = X(q)$ (rationale) function

$$T_q \to \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad S_q \to \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$$

<u>Lemma</u> $\quad S_q^2 = (T_q S_q)^3 = Id.$

Thm (Morier-Genoud V. O.) Uniqe map $[\ ]_q : \mathbb{Q} \cup \{\infty\} \to \mathbb{Z}(q)$

- commuting /w $PSL(2, \mathbb{Z})$-action

- $[0]_q = 0$.

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uniqueness $\Leftarrow$ transitivity of $PSL(2, \mathbb{Z})$-act.

existence $\Leftarrow$ expl. formula

Explicit formula

$$\frac{n}{m} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_k}}}$$

$$\left[\frac{n}{m}\right]_q = [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{\ddots + \cfrac{q^{\pm a_{k-1}}}{[a_k]_{q^{\mp 1}}}}}}$$

Remark: reason why we choose

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}.$$

Euler/Gauss $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$

$$[n+1]_q = q[n]_q + 1 \longleftarrow T_q[n]_q$$

S_q - unique linear-fractional transformation

$$S_q^2 = (T_q S_q)^3 = Id.$$

Another reason: relation to Burau repr. of B_3.

Examples & properties

$$\left[\frac{1}{2}\right]_q = \frac{q}{1+q}$$

$$\left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q} \qquad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

Thm $$\left[\frac{n}{m}\right]_q = \frac{N(q)}{M(q)}$$ ← monic unimodal polynomials

(Conjectured [M-GO], proved Oguz-Ravichandran 2023)

"Total positivity"

$$\frac{n}{m} > \frac{n'}{m'}$$

$$¿ \quad \left[\frac{n}{m}\right]_q = \frac{n(q)}{m(q)} > \left[\frac{n'}{m'}\right]_q = \frac{n'(q)}{m'(q)} \quad ?$$

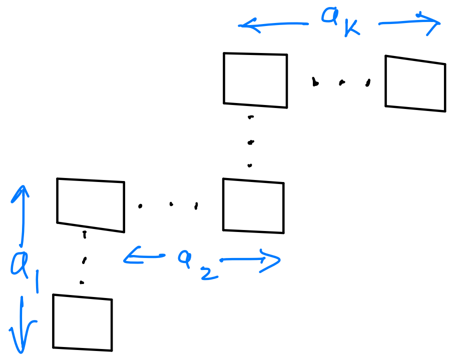
Thm (M-G.O) $\quad n(q)\,m'(q) - m(q)\,n'(q)$

polynomial /w positive integer coeffs.

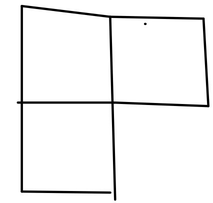
(topological nature!)

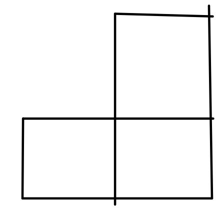
Counting on snake graphs

$$\frac{n}{m} = [a_1, a_2, \dots, a_k]$$



Ex: $\frac{5}{2} = [2, 2]$ $\qquad$ $\frac{5}{3} = [1, 1, 1, 1]$





Thm (Propp ~2000) n = # NE paths in snake graph

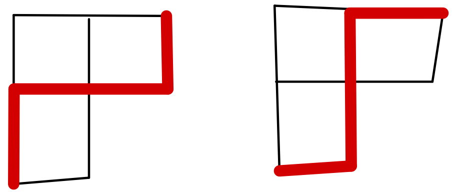
Thm (Ovenhouse '22) (MGO)

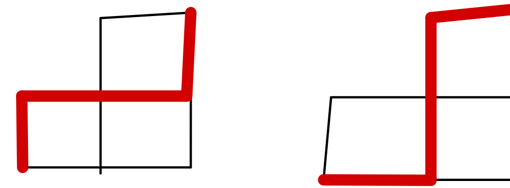
$$\left[\frac{n}{m}\right]_q = \frac{n(q)}{m(q)} \quad ; \quad n(q) = \sum_{k=0}^{\ell} n_k q^k$$

n_k = # NE paths w/ k boxes under path

Examples

$$\begin{bmatrix}5\\2\end{bmatrix}_q = \frac{1+2q+q^2+q^3}{1+q} \qquad \begin{bmatrix}5\\3\end{bmatrix}_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$





Thm (Ovenhouse '22)

Let $q = p^{\ell}$, then $N(q)$ in $\begin{bmatrix}n\\m\end{bmatrix}_q =: \frac{N(q)}{M(q)}$

is # points in $Gr_{r,s}(\mathbb{F}_q)$, where

$$r = a_1 + a_3 + a_5 \cdots$$

$$s = a_1 + a_2 + a_3 + a_4 \cdots$$

4.1 q-irrationals

$x \in \mathbb{R} \setminus \mathbb{Q}$ what is $[x]_q$ - ?

Take x_n sequence of rationals $x_n \to x$

$[x_n]_q \longrightarrow \ldots$??

Thm (i) Taylor series $[x_n]_q = \sum_k \varkappa_{n,k}\, q^k$ stabilizes as n grows

(ii) The result (stabilized series) $\sum \varkappa_k\, q^k$ depends on x only.

Example.

$$\frac{F_{n+1}}{F_n} \leftarrow \text{Fibonacci} \rightarrow \frac{1+\sqrt{5}}{2}$$

$$\left[\frac{8}{5}\right]_q = \frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3} = 1+q^2-q^3+2q^4-4q^5+7q^6-12q^7\ldots$$

$$\left[\frac{21}{13}\right]_q = \frac{1+3q+4q^2+5q^3+4q^4+3q^5+q^6}{1+3q+3q^2+3q^3+2q^4+q^5} = 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+36q^8\ldots$$

$$[\varphi]_q = 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8-82q^9+185q^{10}\ldots$$

A004148 "Generalized Catalan numbers"

Generating f-n

$$[4]_q = \frac{q^2+q-1+\sqrt{(q^2+3q+1)(q^2-q+1)}}{2q}$$

Another example:

$$\left[\sqrt{2}+1\right]_q = \frac{q^2+2q-1+\sqrt{(q^4+q^3+4q^2+q+1)(q^2-q+1)}}{2q}$$

. . .

$q=1$

Quadratic irrationals

<u>Thm</u> (Morier-Genoud-Leclere '21)

$$x = \frac{a + b}{c} \quad \text{then} \quad [x]_q = \frac{A(q) + \sqrt{B(q)}}{C(q)}$$ ← palindromic

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x - fixed point of $A \in PSL(2, \mathbb{Z})$

$$\underbrace{A_q\left([x]_q\right) = [x]_q}$$

5. Hankel determinants

Thm (Pedon, V.O. '24) First 4 Hankel seq. 4-antiperiodic

$$\Delta_n(\varphi) = 1, 1, 1, 0, -1, -1, -1, 0 \, \ldots$$

$$\Delta_n^{(1)}(\varphi) = 1, 0, -1, 1, -1, 0, 1, -1 \, \ldots$$

$$\Delta_n^{(2)}(\varphi) = 1, 1, 1, 0, -1, -1, -1, 0 \, \ldots$$

$$\Delta_n^{(3)}(\varphi) = 1, -1, 0, 0, -1, -1, 0, 0 \, \ldots$$

$[\varphi]_q$ is uniquely determined by this

Similar properties for $1+\sqrt{2} = [2,2,2,\dots]$

$$\frac{3+\sqrt{13}}{2} = [3,3,3,\dots]$$

$$\dots$$

$$y_k = [k,k,k,\dots]$$

More and more

sequences $\Delta(y_k)$, $\Delta^{(1)}(y_k)$ ... $\Delta^{(\ell)}(y_k)$

consist only of $\{-1,0,1\}$ and (anti) periodic

<u>Conjectured</u> (Pedon – V.O)

<u>Proved</u> (Pedon - G. Han '25)

6) Somos & Gale-Robinson recurrences

Observation: First 3 seq. $\Delta_n(\varphi)$, $\Delta^{(1)}_n(\varphi)$, $\Delta^{(2)}_n(\varphi)$ satisfy

$$\boxed{\Delta_{n+4}\Delta_n = \Delta_{n+3}\Delta_{n+1} - \Delta^2_{n+2}}$$

" Somos-4 "

Similarly, for $[1+\sqrt{2}]_n$

$$\Delta_{n+6}\Delta_n = \Delta_{n+5}\Delta_{n+1} - \Delta^2_{n+3}$$

...

Somos - 4:

$$a_{n+4}\, a_n = \alpha\, a_{n+3}\, a_{n+1} + \beta\, a_{n+2}^2$$

Discrete integrable dynamical system!

$(a_0, a_1, a_2, a_3)\ a_4\ a_5\ \ldots$ (Fordy-Hone 2014)

"initial conditions"

Relation to cluster algebras (Fomin, Zelevinsky, Marsh ...)

Same for other Gale-Robinson seq.

7. A naive question

Are Catalan & Motzkin - q-numbers ?

$$C(q) = \frac{1-\sqrt{1-4q}}{2q}$$

$$M(q) = \frac{1-q-\sqrt{(1+q)(1-3q)}}{2q^2} = \frac{1}{q}\, C\left(\frac{q}{1-q}\right)$$

No: $C(q), M(q)$ are not fixed points of $A_q \in PSL(2,\mathbb{Z})$.

But...

$$T_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \quad S_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}; \quad T_q S_q = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Fakt

$$T_q S_q (C(q)) = q\, C(q)$$

$$S_q (M(q)) = q\, M(q) + \frac{q-1}{q}.$$

$q = 1$

$$C(1) = \frac{1+i\sqrt{3}}{2}, \quad M(1) = i \quad \text{- complex!}$$

Fixed TS, S

q - complex numbers??