Elliptic genus of Calabi-Yau varieties and modular differential equations (MDEs) of degree ≥ 3

Valery Gritsenko (a joint project with **Dimitrii Adler**)

ILMS&AF, HSE University, Moscow
Laboratoire Paul Painlevé

Algebraic geometry, integrable systems and automorphic forms
Mai 26, 2025

1. MDEs of modular forms: Number theory, VOAs

1. D. Zagier, Modular forms and differential operators, Proc. Indian Acad. Sci. Math. Sci. 104 (1994) 57–75.

2. M. Kaneko, D. Zagier, Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials, AMS/IP Stud. Adv. Math. 7 (1998) 97–126.

3. M. Kaneko, M. Koike, On modular forms arising from a differential equation of hypergeometric type, Ramanujan J. 7 (2003) 145–164.

4. M. Kaneko, K. Nagatomo, Y. Sakai, The third order modular linear differential equations, J. Algebra 485 (2017) 332–352.

5. Y. Arike, K. Nagatomo, Y. Sakai, Vertex operator algebras, minimal models, and modular linear differential equations of order 4, J. Math. Soc. Jpn. 70 (4) (2018) 1347–1373.

6. K. Kawasetsu, Y. Sakai, Modular linear differential equations of fourth order and minimal w-algebras, J. Algebra 506 (2018) 445–488.

2. MDEs of Jacobi modular forms: Invariant theory, Geometry, String theory

1. T. Kiyuna, Kaneko–Zagier type equation for Jacobi forms of index 1, Ramanujan J. 39 (2016) 347–362.

2. D. Adler, V. Gritsenko, The D_8-tower of weak Jacobi forms and applications, J. Geom. Phys. 150 (2020) 103616.

3. J-W. van Ittersum, G. Oberdieck, A. Pixton, Gromov-Witten theory of K3 surfaces and a Kaneko–Zagier equation for Jacobi forms, Sel. Math. 27 (2021), 64.

4. S. Kimura, The modular differential equation for skew-holomorphic Jacobi forms, Ramanujan J. 59 (2022) 1137–1146.

5. D. Adler, V. Gritsenko, Elliptic genus and modular differential equations, J. Geom. Phys. 181 (2022) 104662.

6. D. Adler, V. Gritsenko, Modular differential equations of the elliptic genus of Calabi-Yau fourfolds, J. Geom. Phys. 194 (2023) 104995.

7. D. Adler, V. Gritsenko, Modular differential equations of $W(D_n)$-invariant Jacobi form, J. Geom. Phys. 206 (2024) 105339.

Let $M = M_d$ be an (almost) complex compact manifold of (complex) dimension d and T_X be its tangent bundle. For $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$ we put $q = e^{2\pi i\tau}$ и $\zeta = e^{2\pi i z}$. One can define a formal series which is *a K-theory variant of Jacobi triple product*.

$$\mathbb{E}_{q,\zeta} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-\zeta^{-1}q^n} T_M^* \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-\zeta q^n} T_M \bigotimes_{n=0}^{\infty} S_{q^n} T_M^* \bigotimes_{n=0}^{\infty} S_{q^n} T_M =$$

$$= \sum_{n,m} E_{n,l} q^n \zeta^l \in K(M)[q,\zeta]$$

where

$$\bigwedge\nolimits_x E = \sum_{k \geqslant 0} (\wedge^k E) x^k, \quad S_x E = \sum_{k \geqslant 0} (S^k E) x^k,$$

and $\wedge^k$ (resp. S^k) is the k^{th} exterior (resp. symmetric) power.

4. Elliptic genus of M_d with $c_1(M) = 0$

We put

$$EG(M) = \chi(M; \tau, z) = \zeta^{\frac{d}{2}} \int_M \mathrm{ch}(\mathbb{E}_{q,\zeta})\mathrm{td}(T_M) = \sum_{n \geq 0,\, l \in \mathbb{Z}} a(n, l)\, q^n \zeta^l$$

where td is *the Todd class*, $\mathrm{ch}(\mathbb{E}_{q,\zeta})$ is *the Chern character* applied to each coefficient of the formal power series. The q^0-part of $\chi(M; \tau, z)$ is (up to a renormalisation) **the Hirzebruch χ_y-genus** of M. Then, by [Kawai-Yamada-Yang 1994], [Gritsenko 1999] and [Totaro 2000] we have

Theorem. *If M_d is a compact complex manifold of dimension d with $c_1(M_d) = 0$ (over $\mathbb{R}$), then its elliptic genus $\chi(M_d; \tau, z)$ is a weak Jacobi form of weight 0 and index $\frac{d}{2}$ with integral Fourier coefficients.*

Theorem [Gr. 1999]. The elliptic genus $\chi(M_d; \tau, z)$ is uniquely determined by its q^0-coefficient or by its Hirzebruch χ_y-genus, if $d < 12$ or $d = 13$.

5. Weak Jacobi forms for A_1

Let be $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$. **A weak Jacobi form of weight** k **and index** m is a holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which satifies :

- $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \varphi(\tau, z)$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$;
- $\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m (\lambda^2\tau + 2\lambda z)} \varphi(\tau, z)$ for $\lambda, \mu \in \mathbb{Z}$;
- the form $\varphi(\tau, z)$ has the following Fourier expansion:

$$\varphi(\tau, z) = \sum_{l \in \mathbb{Z}} \sum_{n \geqslant 0} a(n, l)\, q^n \zeta^l, \quad q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}.$$

The bigraded ring of such Jacobi forms of the even weights is a free algebra in four variables (Eichler-Zagier 1985):

$$J^w_{2*,*} = \bigoplus_{k,m \in \mathbb{Z}} J^w_{2k,m} = \mathbb{C}\left[E_4(\tau), E_6(\tau), \varphi_{0,1}(\tau, z), \varphi_{-2,1}(\tau, z)\right].$$

6. Jacobi modular forms for $A_1 = \langle 2 \rangle$: examples

The odd Jacobi theta-series of characteristic 2 ($\vartheta(-z) = -\vartheta(z)$)

$$\vartheta(\tau, z) = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1 - q^n\zeta)(1 - q^n\zeta^{-1})(1 - q^n)$$

is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with a multiplier system of order 8. The main special functions in our context:

$$\frac{\partial\vartheta(\tau, z)}{\partial z}\Big|_{z=0} = 2\pi i\, \eta(\tau)^3, \quad \frac{\partial^2 \log \vartheta(\tau, z)}{\partial z^2} = -\wp(\tau, z) - \frac{8}{24}\pi^2 E_2(\tau),$$

$$E_2(\tau) = \tfrac{24}{2\pi i} \frac{d\ \log(\eta(\tau))}{d\tau} = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n.$$

The main generators of the ring of weak Jacobi forms are

$$\varphi_{-2,1}(\tau, z) = \frac{\vartheta^2(\tau, z)}{\eta^6(\tau)} = (\zeta - 2 + \zeta^{-1}) + q \cdot (\ldots) \in J_{-2,1},$$

$$\varphi_{0,1}(\tau, z) = -\tfrac{3}{\pi^2}\wp(\tau, z)\varphi_{-2,1}(\tau, z) = (\zeta + 10 + \zeta^{-1}) + q \cdot (\ldots).$$

7. SL_2-modular differential operator $\mathbb{D}_k$ on M_k

We put $\mathbb{D} = 12q\frac{d}{dq} = 6\pi i \frac{d}{d\tau}$. For any automorphic function $f(\tau)$ of weight 0 one gets a form of weight 2: $\mathbb{D}(f) \in M_2^{(mer)}(SL_2(\mathbb{Z}))$. Therefore, passing through weight zero, we have

$$\mathbb{D}_k : M_k(SL_2(\mathbb{Z})) \to M_{k+2}(SL_2(\mathbb{Z})), \quad \mathbb{D}_k(f) = 12D(f) - kE_2 \cdot f,$$

where $E_2(\tau)$ is the quasi-modular Eisenstein series from page 6.

Ramanujan's system for generators of $M_*^{quasi} = \mathbb{C}[E_2, E_4, E_6]$

$$\mathbb{D}(E_2) = E_2^2 - E_4, \;\; \mathbb{D}_4(E_4) = -4E_6, \;\; \mathbb{D}_6(E_6) = -6E_4^2.$$

We add the **Chazy equation** for the quasi-modular Eisenstein series $E_2(\tau)$: $y''' = 2yy'' - 3(y')^2$.

Kaneko-Zagier equation (for characters of vertex algebras):

$$f''(\tau) - \tfrac{k+1}{6}E_2(\tau)f'(\tau) + \tfrac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0,$$

$$\mathbb{D}_{k+2} \circ \mathbb{D}_k(f) - k(k+2)E_4 \cdot f = 0.$$

8. The heat operator H and the MDO H_k of weight k

We can change the operator $\mathbb{D}$ by a renormalided **heat operator**

$$H^{(m)} = \frac{3}{m}\frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right) = 12q\frac{d}{dq} - \frac{3}{m}\left(\zeta\frac{d}{d\zeta}\right)^2$$

$$H^{(m)}(a(n,l)q^n\zeta^l) = \frac{3}{m}(4nm - l^2)q^n\zeta^l.$$

As for $\mathbb{D}_k$ we get a modular correction of $H = H^{(m)}$:

$$H_k : J^w_{k,m} \to J^w_{k+2,m}, \quad H_k(\varphi_{k,m}) = H(\varphi_{k,m}) - \frac{(2k-1)}{2}E_2\cdot\varphi_{k,m}.$$

We have $H_{\frac{1}{2}}(\vartheta(\tau,z)) = H(\vartheta(\tau,z)) = 0$, $H_{-2}(\varphi_{-2,1}) = (-\frac{1}{2})\varphi_{0,1}$, and an analogue of the **Kaneko-Zagier equation** for $\varphi_{-2,1}$ is true

$$H_0 \circ H_{-2}(\varphi_{-2,1}) - \frac{5}{4}E_4\varphi_{-2,1} = 0.$$

9. MDE for the elliptic genus of a K3 surface

For $CY_2 = K3$ we have

$$EG(K3;\tau,z) = 2\varphi_{0,1}(\tau,z) = 2(\zeta^{-1} + 10 + \zeta) + 2q(\ldots).$$

If $H_0^{[3]} = H_4 \circ H_2 \circ H_0$, then according to [AG2022] we have

$$H_0^{[3]}(\varphi_{0,1}) - \tfrac{101}{4} E_4 H_0(\varphi_{0,1}) + 10 E_6 \varphi_{0,1} = 0$$

or

$$(H^3 - \frac{9}{2} E_2 H^2 + \frac{9}{2}(6E_2' - 5E_4)H + 27(E_2'' - \frac{5}{4}E_4'))(\varphi_{0,1}) = 0.$$

We call such an equation **a modular differential equation** (or **MDE**) of degree 3. For CY_5 we get another MDE of degree 3

$$(H_0^{[3]} - \tfrac{611}{25} E_4(\tau) H_0 + \tfrac{88}{25} E_6(\tau))(EG(CY_5)) = 0.$$

A Jacobi analogue of the **Kaneko-Zagier equation** is the following MDE of degree 2

$$H_{k+2} \circ H_k(\varphi_{k,m}) = \lambda E_4(\tau) \varphi_{k,m}.$$

The characters of vertex operator algebras satisfy a KZ equation. The Jacobi form $\varphi_{0,1}$ is *the partition function* of multiplicities of the positive roots of a Lorentzian Kac-Moody Lie algebra with three real simple roots. Another example is a nearly holomorphic Jacobi form

$$\psi_{0,1}^{(-1)}(\tau, z) = q^{-1} + (\zeta^{\pm 2} + 70) + q(\dots)$$

which determines a Lorentzian Kac-Moody algebra with the simplest possible Cartan matrix of the three real simple roots from the Gritsenko-Nikulin list. Then

$$(H_0^{[3]} - \tfrac{533}{4} E_4 H_0 + 874 E_6)(\psi_{0,1}^{(-1)}) = 0.$$

The Fourier coefficients $a(n, l)$ of the elliptic genus of a K3 surface satisfy the following recurrence relation

$$(4n - l^2)\left((4n - l^2)(4n - l^2 - \tfrac{3}{2}) - \tfrac{5}{2}\right) a(n, l) =$$

$$24 \sum^{n} a(n-s, l)\left[(4n-4s-l^2)(3s\sigma_1(s)+25\sigma_3(s))+s^2\sigma_1(s)+\tfrac{25}{2}s\sigma_3(s)\right].$$

Theorem. EG&MDE$_3$. [Adler-Gr. 2022, 2023, 2025].

1. The elliptic genus of a Calabi-Yau threefold satisfies a MDE of degree 1: $H(EG(CY_3)) + \frac{1}{2}E_2(\tau) \cdot EG(CY_3) = 0.$

2. The elliptic genus of a Calabi-Yau variety CY_{2d} of dimension $2d$ in strict sense ($h^{p,0}(CY_{2d}) = 0$ for every $0 < p < 2d$) does not satisfy a MDE of degree 2.

3. The elliptic genus of the following Calabi-Yau varieties satisfies a MDE of degree 3:

- a K3 surface and any CY_5 of dimension 5,
- a CY_4 of dimension 4 with $e(CY_4) = 48$ or -18.

4. For all other Calabi-Yau variety CY_{2d} (in strict sense) of dimension $2d < 12$ its elliptic genus **does not satisfies** a MDE of degree 3.

12. CY_4 with e=48: an automorphic analogue of K3

Application. The $EG(CY_4)$ with $e(CY_4) = 48$ looks very similar to K3: $\quad EG(CY_4^{e=48}) = \psi_{0,2}(\tau, z) = 2(\zeta^{-2} + 22 + \zeta^2) + q(\dots)$.

1) $\psi_{0,2}$ satisfies a MDE of degree 3 without the Eisenstein series E_6 in its quasi-modular version:

$$\mathbf{K3} : \left(H^3 - \frac{9}{2}E_2H^2 + \frac{9}{2}(6E_2' - 5E_4)H + 27(E_2'' - \frac{5}{4}E_4')\right)(\varphi_{0,1}) = 0,$$

$$\mathbf{CY}_4^{e=48} : \left(H^3 - \frac{9}{2}E_2H^2 + 9(3E_2' - 7E_4)H + 27(E_2'' - \frac{7}{2}E_4')\right)(\psi_{0,2}) = 0.$$

2) EG(K3) and EG($CY_4^{e=48}$) are generating functions of the multiplicities of all positive roots of Lorentzian Kac-Moody algebras with three real simple roots in Gritsenko-Nikulin's classification.

3) SQEG$(\mathrm{K3})^{-1}$ и SQEG$(CY_4^{e=48})^{-1}$ are two basic Siegel cusp forms with respect to the Siegel paramodular groups of genus two $\Gamma_1 = Sp_4(\mathbb{Z})$ and Γ_2.

To prove the claim **4** of **Theorem EG&MDE$_3$** we analysed carefully the cases of Jacobi forms of weight 0 and index 2 and 3 in order to calculate explicit MDEs of $EG(M_4)$ and $EG(M_6)$ for any M_{2d} with $c_1(M_{2d})=0$.

Theorem [AG-2023]. **The case of dimension 4.**

1. A generic weak Jacobi form of weight 0 and index 2 satisfies a MDE of degree 5.

2. Let K_4 and A_4 be hyperkähler varieties of type $\mathrm{Hilb}^2(K3)$ and $\mathrm{Kum}^2(A)$. Their elliptic genera satisfy MDEs of degree 5

$$\left(H_0^{[5]}-\frac{815}{6}H_0^{[3]}+\frac{1885}{4}E_6H_0^{[2]}+\frac{99455}{48}E_4^2H_0-\frac{20845}{48}E_4E_6\right)(EG(K_4))=0,$$

$$\left(H_0^{[5]}-\frac{13775}{106}E_4H_0^{[3]}+\frac{114865}{212}E_6H_0^{[2]}+\frac{1848045}{848}E_4^2H_0-\frac{381975}{848}E_4E_6\right)(EG(A_4))=$$

3. There are **three** weak Jacobi forms in $J_{0,2}^w$ satisfying MDEs of degree 3, **one** Jacobi form satisfying MDE of degree 4 and **one** Jacobi form satisfying MDE of degree 6.

Theorem [AG-2025]. Let $\varphi_{k,m} \in J^{w}_{k,m}$ be a solution of a MDE of degree 3

$$(H_k^{[3]} + \lambda E_4 H_k + \mu E_6)(\varphi_{k,m}) = 0.$$

Then $q^0(\varphi_{k,m})$ does not contain **four** non-zero Fourier coefficients $a(0,l)$ with pairwise different l^2. If it contains exactly **three** such coefficients, then

1) $l_1^2 + l_2^2 + l_3^2 = -\frac{2k+1}{2} m$;

2) The weight k is negative and $k \geq -5$;

3) The q^0-part of the Fourier expansion of $\varphi_{k,m}$ has the following form for even and odd k

$$q^0[\varphi] = \begin{cases} a\zeta^{l_1} + a\zeta^{-l_1} + b\zeta^{l_2} + b\zeta^{-l_2} - (a+b)\zeta^{l_3} - (a+b)\zeta^{-l_3}, \\ a\zeta^{l_1} - a\zeta^{-l_1} + b\zeta^{l_2} - b\zeta^{-l_2} + c\zeta^{l_3} - c\zeta^{-l_3}. \end{cases}$$

Corollary. *For* $2 \leq d \leq 5$ *there are only two Jacobi forms in* $J^{w}_{0,d}$ *which might represent* $EG(CY_{2d})$ *and satisfy* MDE_3.

15. Varieties of dimension 6

Theorem [AG-2025]. **The case of $J^w_{0,3}$ or $EG(M_6)$.**

1) A generic $\varphi \in J^w_{0,3}$ satisfies a MDE of degree 7 ([AG-2022]).

2) In $J^w_{0,3}$ there are no solutions of MDEs of degree 1, 2 or 3.

3) There are 10 Jacobi forms that satisfy MDEs of degree 4. Between them there are three candidates which might be equal to the elliptic genus of a CY_6 with fixed χ_y-genus.

4) There are 5 Jacobi forms that satisfy MDEs of degree 5 and only one of them might be the elliptic genus of a CY_6.

5) There are exactly 4 Jacobi forms that satisfy the MDE of degree 6. For special parameters they might satisfy the MDE of degree 8.

Corollary. *The minimal possible degree of MDE for $EG(CY_6)$ is 4.*

To get a similar theorem for $2d \geq 8$ one needs much more calculations! To finish the proof of Theorem EG&MDE$_3$ for strict CY_{2d} for $2d = 8, 10$ we use the Corollary from p.14.

In our method we analyse only the polynomial part of q^0-part of the Fourier expansion, i.e. we use **the invariant theory** to find particular solutions of MDEs. For example,

$$\vartheta^2(\tau,z),\ \vartheta^3(\tau,z),\ \vartheta^2(\tau,z),\ \vartheta(\tau,2z)\quad \text{satisfy KZEs or MDE}_2,$$

$$\vartheta^4,\ \vartheta^5\quad \text{satisfy MDE}_3,$$

$$\vartheta^6,\ \vartheta^7\quad \text{satisfy MDE}_4,\qquad \vartheta^8\quad \text{satisfies MDE}_5.$$

After multiplication by powers of $\eta(\tau)$ with $z \to mz$ we get many solutions of KZEs: $\quad H_{k+2}\circ H_k(\varphi_{k,m}) = \lambda E_4(\tau)\varphi_{k,m}$.

$$\varphi_{10,1}(\tau,z)=\eta^{18}(\tau)\vartheta^2(\tau,z),\ \varphi_{9,3}(\tau,z)=\eta^{15}(\tau)\vartheta^2(\tau,z)\vartheta(\tau,2z)\ (\lambda=\frac{5}{4}),$$

$$\varphi_{9,6}(\tau,z)=\eta^{15}(\tau)\vartheta^3(\tau,2z)\quad(\lambda=3),$$

$$\varphi_{10,10}(\tau,z)=\eta^{18}(\tau)\vartheta(\tau,2z)\vartheta(\tau,4z)\quad(\lambda=\frac{11}{25}).$$

Dimitrii Adler found **infinite towers** of solutions based on such weak Jacobi forms.