Bessel and beyond.
(Old songs with new motives.)

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"Algebraic geometry, integrable systems and automorphic forms" (AGISAF-2025)
Laboratoire Paul Painlevé, Lille, 26-30 Mai 2025

based on:

(arXiv: 2102.09511 with V. Golyshev, A. Mellit, and D. van Straten)
(arXiv: 2405.03015 with Ilia Gaiur and D. van Straten, to appear in "Selecta...")

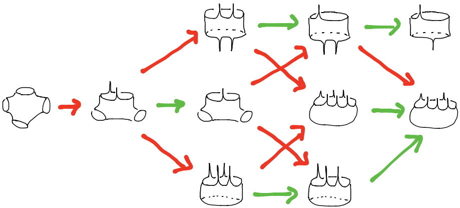
Plan of talk

1. Objects, origins, and motivations.
2. $N-$Bessel kernels : basics
3. N-Bessel kernels as periods
4. Singularities: Landau Discriminants, Buchstaber-Rees
5. Integrality properties: computer experiments
6. Links and perspectives

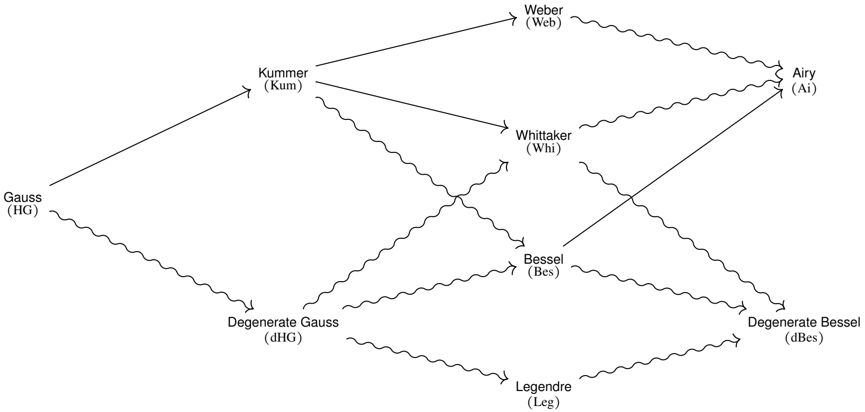
Confluence of Painlevé equations:

First diagram - H. Sakai, Second diagram- after L. Chekhov, M. Mazzocco and V.R

$$\begin{tikzcd}
 & & P_{III}^{D_6} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[r] \arrow[ur] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[r] \arrow[ur] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$



Confluence of Hypergeometry (after K. Iwaki and O. Kidwai):



Confluence of Painlevé and Hypergeometry (after O. Lisovyy and P. Gavrylenko):

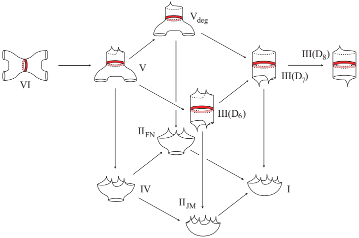


Figure 3: CMR confluence diagram for Painlevé equations.

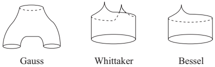


Figure 4: Some solvable RHPs in rank $N = 2$: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank $\frac{1}{2}$).

Objects, origins, and motivations

Main Heroes

- $N-$ Bessel $=$ "$\mathbb{P}^{N-1}-$Quantum DO:

$$\boxed{\left(x\frac{d}{dx}\right)^N \Psi - \lambda^N x\Psi = 0, \quad \lambda \in \mathbb{C}}$$

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- Analytic solution:

$$\boxed{\sum_{k\geq 0} \frac{(\lambda^N x)^k}{(k!)^N} = \frac{1}{(2\pi\sqrt{-1})^{N-1}} \oint_{(S^1)^{N-1}} \exp\lambda\left(\sum_{k=1}^{N-1} z_k + \frac{x}{\prod_{k=1}^{N-1} z_k}\right)\prod_{k=1}^{N-1}\frac{dz_k}{z_k}}$$

Bessel Equation(s)

- Classical Bessel:

$$\boxed{x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}-(x^2+\nu^2)f=0,\ \nu\in\mathbb{R},\ J_\nu(x)=\sum_{k\geq 0}\frac{(-1)^k}{\Gamma(\nu+1+k)k!}\left(\frac{x}{2}\right)^{\nu+2k}}$$

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- Modified Bessel:

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$$\boxed{\theta^2u=\lambda^2xu,\quad \theta=x\frac{d}{dx}.}$$

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- $(N=2,\lambda=-1.$ Solutions)

$$\boxed{J_0(2\sqrt{-x})=I_0(2\sqrt{x})=\sum_{k=0}^{\infty}\frac{x^k}{k!^2}=\frac{1}{2\pi\sqrt{-1}}\oint e^{-\left(z+\frac{x}{z}\right)}\frac{dz}{z}={}_0F_1\left(-;1\mid x\right)}$$

Origins-1

Gauss → Kummer → Bessel: confluence

- Gauss hypergeometry

$$\boxed{\frac{d^2F}{dx^2} + \frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dF}{dx} - \frac{\alpha\beta}{x(1-x)}F = 0,\ \gamma \notin (-\mathbb{N}),\ F(x) = {}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\ \Big|\ x\right]}$$

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- From Gauss to Kummer

$$\boxed{x\frac{d^2F}{dx^2}+(\gamma-x)\frac{dF}{dx}-\alpha F=0,\ \beta=\epsilon, x=\frac{x}{\epsilon},\ \epsilon\to\infty, F(x)={}_1F_1\left[\begin{matrix}\alpha\\ \gamma\end{matrix}\mid x\right]}$$

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- From Kummer to Bessel

$$\boxed{F(y):=y^{-\nu}\exp(iy)f(y), \alpha:=\nu+1/2, \gamma:=2\nu+1,}$$

$$\boxed{y\frac{d^2F}{dy^2}+(\gamma-2\sqrt{-1}u)\frac{dF}{dy}-\alpha F=0\Rightarrow y^2\frac{d^2f}{dy^2}+y\frac{df}{dy}+(y^2-\nu^2)f=0}$$

Interpretation of $N-$Bessel on the language of algebraic $\mathcal{D}-$modules

- $N-$Bessel equation can be converted to a connection Be_n on the rank n trivial bundle on the multiplicative group $\mathbb{G}_{m,\mathbb{C}}$.

$$\nabla = x\frac{d}{dx} + A = x\frac{d}{dx} + \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda^n x & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

(the Bessel connection of rank N.)

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- The singularity at $x = \infty$ is irregular with pure slope $\frac{1}{N}$ and the eigenvalues

$$\nu_k(x) = \exp(\frac{2\pi\sqrt{-1}k}{N})x^{\frac{1}{N}}.$$

Mirror of $\mathbb{P}^1$

The integral

$$I = \int_{\gamma\subset\{y_1y_2=q\}} \exp\frac{(y_1+y_2)}{\hbar}\frac{dy_1dy_2}{dq}$$

satisfies the Bessel differential equation

$$[\hbar\theta_q)]^2 I = qI$$

and therefore (Y_q, f_q, ω_q) where Y_q is given by the equation $y_1y_2 = q$ in $Y = \mathbb{C}^2$, the function f_q is the restriction to Y_q of $f = y_1 + y_2$, and ω_q is the relative "volume" form $\frac{(dy_1dy_2)}{d(y_1y_2)}$ on Y_q can be taken on the role of the mirror partner of $X = \mathbb{P}^1$.

Number Theory analogy:

- $N = 2$ Kloostermann solution

$$J^{(2)} := I_0(2\sqrt{\lambda x}) = \sum_{k=0}^{\infty} \frac{\lambda x^k}{k!^2} = \frac{1}{2\pi\sqrt{-1}} \oint e^{\lambda\left(z+\frac{x}{z}\right)} \frac{dz}{z}$$

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$$\mathrm{Kl}(2,a) := \sum_{x\in\mathbb{F}_p^*} \left(\frac{2\pi\sqrt{-1}}{p}\right) \exp\left(x + \frac{a}{x}\right).$$

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- In general, the Kloosterman sum is defined for an integer $N \geq 2$, a prime p and $a \in \mathbb{F}_p^*$ by

$$\mathrm{Kl}(n,a) := \sum_{z\in\mathbb{F}_p^*} (\frac{2\pi\sqrt{-1}}{p}) \exp\left(x_1 + \ldots + x_{N-1} + \frac{a}{x_1 x_2 \ldots x_{N-1}}\right).$$

Origins and motivations-1

Classical Analysis and Special functions:

- Clausen duplication formula ("Crelle", 1828)

$$\left({}_2F_1\left[\begin{matrix} a,b \\ a+b+\frac{1}{2} \end{matrix} \mid x\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a & a+b & 2b \\ & a+b+\frac{1}{2} & 2b \end{matrix} \mid x\right]$$

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-homogeneous commutative coalgebra $C(\mathbb{R}^n)^{O(n)}$.(L. Schläfli (1876), *Math.Ann.*X)
-$n = 2-$ 2-valued cyclic formal group

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$$J_0(x)J_0(y) = \frac{1}{2\pi}\int_{|x-y|}^{x+y} \frac{J_0(z)zdz}{\sqrt{x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2}}, \; x < y \in \mathbb{R}$$

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- $D2$**-equation (small Apèry)**. Algebraic kernel

$$[\partial_t \circ f(t) \circ \partial_t + t]\,\psi = \lambda\psi, \quad f(t) = t^3 + At^2 + t$$

$$\phi_\lambda(x)\phi_\lambda(y) = \int \frac{\phi_\lambda(z)}{\sqrt{P(x,y,z)}}dz, \quad P(x,y,z) = \Delta_t\left[f(t) - (t-x)(t-y)(t-z)\right]$$

  An appearance of **Kontsevich polynomials**

$N-$Bessel kernels : basics

N-Bessel

Consider the $N-$ Bessel equation

$$\left(x\frac{d}{dx}\right)^N \Psi - x\lambda\Psi = 0$$

and its analytic solution ($N-$ Bessel function)

$$\Psi(x,\lambda) = J^{(N)}(\lambda;x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k!)^N} x^k$$

Set $\lambda = 1$.
The Laplace transform:

$$\mathcal{L}(J^{(N)})(z) = \sum_{k=0}^{\infty} \frac{(Nk)!}{k!^N} z^k$$

$$J^{(N)}(x)J^{(N)}(y) = \sum_{n,m} \left(\frac{1}{n!m!}\right)^N x^n y^m = \sum_{n,m} \binom{n+m}{n}^N x^n y^m \left(\frac{1}{(n+m)!}\right)^N$$

Integral form

The solution can be written in the integral form

$$J^{(N)}(x) = \frac{1}{(2\pi i)^{N-1}} \oint \exp\left(\sum_{i=1}^{N-1} Y_i + \frac{x}{Y_1 Y_2 \ldots Y_{N-1}}\right) \frac{dY_1}{Y_1}\frac{dY_2}{Y_2} \ldots \frac{dY_{N_1}}{Y_{N-1}}, \quad (1)$$

where one may recognize a Dwork family of super-potentials in the exponential function.

N-Bessel-2

When we multiply this by $z^{-(m+n)}$, multiply with $J^{(N)}(z)$, expand in powers of z we find:

$$J^{(N)}(x)J^{(N)}(y) = \frac{1}{2\pi i}\oint K(x,y,z)J^{(N)}(z)\frac{dz}{z} = \frac{1}{2\pi i}\oint H\left(\frac{x}{z},\frac{y}{z}\right)J^{(N)}(z)\frac{dz}{z},$$

where

$$H(X,Y) = \sum_{n,m}\binom{n+m}{n}^N X^n Y^m.$$

Try to understand the singular locus of the $D-$ module generated by H. For $N=2$ we have the algebraic kernel $H(X,Y) = \frac{1}{\sqrt{\Delta_2(X,Y,1)}}$ and $\Delta_2(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$. This defines a circle, tangent the coordinate lines $xyz=0$ at the mid-points $(0:1:1), (1:0:1), (1:1:0)$ in $\mathbb{P}^2$.

$$\Delta_2(u^2,v^2,w^2) = (u+v+w)(-u+v+w)(u-v+w)(u+v-w) = 16S_\Delta^2-$$

(**"Heron configuration"**)

Duplication kernels

Consider a "diagonal" $x = y$ in order to get one dimensional families

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Convolution of power series and explicit numbers

Duplication kernels

Integral transformation to Sym^2 of DE

$$\Phi_N(x)^2 = \frac{1}{2\pi i} \oint K_N(x/z)\Phi_N(z)\frac{dz}{z}$$

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For $N = 2$ – Clausen duplication

$$\Phi_2(x)^2 = \frac{1}{2\pi i} \oint_T \frac{1}{\sqrt{1 - 4x/z}} \Phi_2(z)\frac{dz}{z}.$$

which leads to

$$(\theta_x^3 - 4x\,\theta_x - 2x)\Phi_2(x)^2 = 0, \quad \theta_x = \frac{xd}{dx}$$

Duplication kernels via multiplicative convolution

Taking the "diagonal" $y = x$ we obtain the Clausen duplication formula

$$\Phi_N(x)^2 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!^N} \right)^2 = \sum_{n=0}^{\infty} \frac{c_n}{n!^N} x^n, \quad c_n = \sum_{k=0}^{n} \binom{n}{k}^N$$

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Hadamard product: $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$

$$(f * g)(x) = \frac{1}{2\pi i} \oint f(x/t) g(t) \frac{dt}{t} = \sum_{j=0}^{\infty} (a_j \cdot b_j) x^j.$$

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Then

$$\Phi_N(x)^2 = \Phi_N(x) \star \sum_{n=0}^{\infty} c_n x^n = \Phi_N(x) \star K(x) = \frac{1}{2\pi i} \oint K(x/t) \Phi_N(t) dt/t$$

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$K(x/z) = K(x, x|z)$ is a generating function of c_n

Binomial coefficient sums

$$K_N(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k}^N \right] t^n, \quad t = x/z$$

- $N = 2$ - Clausen duplication for Bessel

$$K_2(x) = \frac{1}{\sqrt{1-4t}}$$

- $N = 3$ - Apéry like sequence of type A (D. Zagier). Heun function

$$t(t+1)(8t-1)K'' + (24t^2 + 14t - 1)K' + (8t+2)K = 0$$

- Generic $N > 1$: period from Landau-Ginzburg model

$$V_N = \prod_{i=1}^{N-1}(1+Y_i) + \prod_{i=1}^{N-1}(1+Y_i^{-1})$$

- Corresponding family and its period

$$V_N = 1/t, \quad K_N(t) = \frac{1}{(2\pi i)^{N-1}} \oint_{T^{N-1}} \frac{1}{1-tV_N} \frac{dY_1}{Y_1} \frac{dY_2}{Y_2} \cdots \frac{dY_{N-1}}{Y_{N-1}}$$

Picard-Fuchs operators

$$
\begin{aligned}
N=2 \quad & (4t-1)\,\theta_t+2t,\\
N=3 \quad & (t+1)\,(8t-1)\theta_t^2+t\,(16t+7)\,\theta_t+2t(4t+1),\\
N=4 \quad & (16t-1)(4t+1)\theta_t^3+6t(32t+3)\theta_t^2+2t(94t+5)\theta_t+2t(30t+1),\\
N=5 \quad & (32t-1)(4t-7)^2(t^2-11t-1)\theta_t^4+2t(4t-7)(256t^3-2084t^2+4942t+143)\theta_t^3+\\
& t(3072t^4-23024t^3+72568t^2-102261t-1638)\theta_t^2+\\
& t(2048t^4-12896t^3+30072t^2-66094t-637)\theta_t+\\
& 2t(256t^4-1472t^3+1904t^2-7868t-49),\\
N=6 \quad & (t-1)(27t+1)(64t-1)(75t^3+1420t^2+561t+9)\theta_t^6+\\
& +(842400t^6+18022725t^5-1363487t^4-4622791t^3-127551t^2-1977t-9)\theta_t^5+\\
& +5t(452880t^5+10962507t^4+2491544t^3-1779376t^2-46584t-168)\theta_t^4+\\
& +5t(644760t^5+17135271t^4+6994741t^3-1716533t^2-56024t-96)\theta_t^3+\\
& +2t(1282500t^5+36696915t^4+19164721t^3-2088858t^2-100236t-72)\theta_t^2+\\
& +2t(540900t^5+16436910t^4+9826066t^3-428487t^2-38811t-9)\theta_t+\\
& +12t^2(15750t^4+503175t^3+327205t^2-1845t-1044).
\end{aligned}
$$

Picard-Fuchs operators

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ord $\mathcal{D}_{\mathcal{N}}$	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40

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Number of unipotent blocks increase on each fifth iteration

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$$M_0^{(7)} \sim \begin{pmatrix} 1 & 1/2! & 1/3! & 1/4! & 1/5! & 1/6! & 0 & 0 \\ 0 & 1 & 1/2! & 1/3! & 1/4! & 1/5! & 0 & 0 \\ 0 & 0 & 1 & 1/2! & 1/3! & 1/4! & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2! & 1/3! & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/2! & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2! \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Mirror maps

$$N=3 \quad q-3q^2+3q^3+5q^4-18q^5+15q^6+24q^7-75q^8+57q^9+86q^{10}+O(q^{11})$$

$$N=4 \quad q-4q^2-6q^3+56q^4-45q^5-360q^6+894q^7+960q^8-6951q^9+4660q^{10}+O\left(q^{11}\right)$$

$$N=5 \quad q-5q^2-40q^3+115q^4-645q^5-12846q^6-177350q^7- \\ -2574585q^8-44198680q^9-736554815q^{10}+O\left(q^{11}\right)$$

$$N=6 \quad q-6q^2-135q^3-380q^4-24960q^5-696366q^6- \\ -26153302q^7-901888104q^8-35369115894q^9-1381135576280q^{10}+O\left(q^{11}\right)$$

$$N=7 \quad q-7q^2-371q^3-4543q^4-378637q^5-20096783q^6-1568975093q^7 \\ -112310305031q^8-9251250532328q^9-758736375700793q^{10}+O\left(q^{11}\right)$$

N-Bessel kernels as periods

$K_N(x, y|z)$ are periods of $\mathbb{P}^2$-families of algebraic varieties

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$K_N^{(m)}(x_1, x_2 \dots x_m|z)$ are periods of $\mathbb{P}^m$-families

Theorem

Let

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

Define

$$K_N(x,y,z) = \frac{1}{(2\pi i)^{N-1}}\oint \frac{1}{W_N(x,y,z|\mathbf{Y})}\prod_{j=1}^{N-1}\frac{dY_j}{Y_j}$$

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Proof: Kontsevich-Odesskii formal power series for kernel + combinatorics

$$K_N(x,y,z) = \sum_{j,k}\binom{j+k}{k}^N \frac{x^jy^k}{z^{j+k}}.$$

Multi-Kernels

Multi-product

$$\Phi_N(x_1)\Phi_N(x_2)\ldots\Phi_N(x_{m-1})\Phi_N(x_m) = \frac{1}{2\pi i}\oint K_N(x_1,x_2,\ldots x_m\,|z)\,\Phi_N(z)\frac{dz}{z}.$$

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$$W_N^{(m)}(x_1,\dots x_m,z) = \prod_{j=1}^{N-1}\left(1+\sum_{l=1}^{m-1}Y_j^{(l)}\right)\cdot\left(x_1+\sum_{l=1}^{m-1}\prod_{j=1}^{N-1}\frac{x_{l+1}}{Y_j^{(l)}}\right)-z=0,$$

i.e.

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Proof: Similar to the multiplication one

Singularities: Landau Discriminants, Buchstaber-Rees

Appeared families have a precise geometry of singularity loci

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For standard multiplication formulas all singularities are of discriminant type

Product formulas singularities

The singularities of the family

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

is a projective curve in $\mathbb{P}^2$ which a union of triangle

$$xyz = 0$$

and irreducible rational curve

$$\Delta_N(x,y,z) = x^N + y^N + z^N + \dots,$$

which is given by the property

$$\Delta_N(u^N, v^N, w^N) = \prod_{\omega,\eta}(u + \omega v + \eta w),$$

where in the product ω and η run over the N-th roots of unity.

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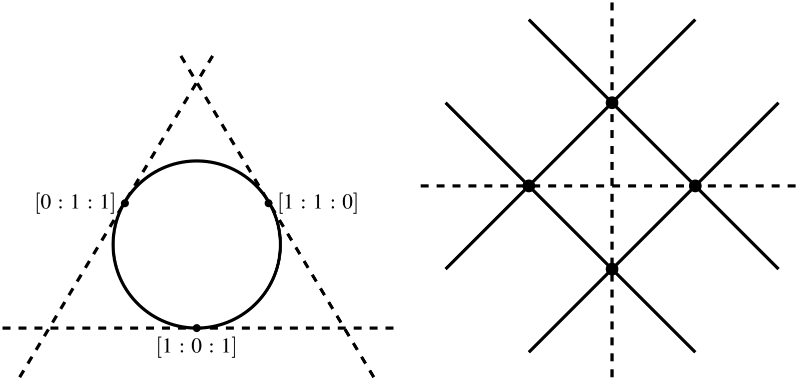
Remark: $\Delta_N(x,y,z)$ us a homogeneous polynomial with integer coefficients

$N = 2$ singularity loci and unfolding



Singular locus

Theorem The polynomials $\Delta_N(x,y,z)$ may be expressed via T-discriminants of the $2N-2$ degree polynomials given by

$$P_{x,y,z}(T) = xT^{N-1}(1+T)^{N-1} + y(1+T)^{N-1} - T^{N-1}z.$$

More precisely, the equality holds

$$(-1)^N (N-1)^{2(N-1)} (xyz)^{N-2} \Delta_N(x,y,z) = \mathrm{disc}_T(P_{x,y,z}(T))$$

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Theorem The projective curve $\Delta_N = 0$ is a rational curve that has $(N-1)(N-2)/2$ double points. All singularities belong to $\mathbb{P}^2(\mathbb{R})$. Moreover, coordinates of the singularities belong to $\mathbb{Q}(\cos(2\pi/N))$.

Links and perspectives

Hecke operators and Kontsevich polynomial

Kontsevich (2007-2009): To make explicite Langlands correspondence for SL_2- local systems L on $\mathbb{P}^1_{\mathbb{F}_p} \setminus \text{four} \quad \text{points}$ with unipotent local monodromies, which correspond to a special Heun equation

$$L := \partial f \partial + t + \lambda, \quad f = t^3 + at^2 + bt + c$$

- a cubic polynomial (See also Teruji Thomas, 2006 (MSci at U Chicago with V. Drinfeld and Niels uit de Bos, 2019 (PhD U Essen with J. Heinloth.)
Define

$$P(x,y,z) = \mathrm{disc}_t(f(t) - (t-x)(t-y)(t-z))$$

Let $\mathbb{F}_p-$ a finite field with a prime characteristic $p \neq 2$. $x \in \mathbb{P}^1(\mathbb{F}_p)$, $H_x \in \mathrm{Mat}_{(p+1)\times(p+1)}(\mathbb{F}_p)$ such that

$$(Hx)_{yz} := 2 + \sharp\{w | w^2 = P(x,y,z) + \text{correction} \quad \text{term}\}$$

A miracle: $[H_x, H_y] = 0$! Operators $H_x, x \in \mathbb{P}^1(\mathbb{F}_p)$ generate *a Hecke algebra* $\mathcal{H}$.

$$\mathcal{H} \curvearrowright \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_p)) : (H_x(\bar{v}))_y = \sum_z (H_x)_{yz}(v_z),\ x,y,z \in P^1(\mathbb{F}_p), (\bar{v}) = (v_x) \in \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_p)).$$

Drinfeld: when $PGL_2(\mathbb{F}_p(t))-$ automorphic representations correspond to normalized common Hecke eigenvectors:

$$H_x(\bar{v}) = v_x\bar{v},\ v_x v_y = \sum_z (H_x)_{yz}(v_z)$$

Hecke operators and Kontsevich polynomial

2-Bessel kernel singularities is a degeneration of the Kontsevich polynomial

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Kernel may be considered as an analog of the Hecke operator in the analytic context

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Higher Bessel Kernels should correpond to the degeneration of the higher rank local systems

Non-Abelian Abel's theorem

- Non-Abelian Abel's theorem by V. Golyshev, V.R., A. Mellit and D. van Straten:

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- N-Bessel kernels lift Buchstaber-Rees law
- Lift from the corresponding spectral curve

$$\delta = x\frac{d}{dx} - A = x\frac{d}{dx} - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ x & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\boxed{\Gamma(x,\lambda) = (-1)^N \det(A/x - \lambda) = \lambda^N - 1/x = 0}$$

Bessel moments

- Bessel moments

$$(\mathcal{P}_k)_{ij} = \int_0^\infty I_0(z)^i K_0(z)^{k-i} z^{2j-1} dz, \quad 1 \leq i,j \leq \lfloor (k-1)/2 \rfloor$$

- Bernoulli matrix

$$(\mathcal{B}_k)_{ij} = (-1)^{k-i} \frac{(k-i)!(k-j)!}{k!} \frac{B_{k-i-j+1}}{(k-i-j+1)!}$$

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$$\mathcal{P}_k \cdot \mathcal{D}_k \cdot \mathcal{P}_k = (-2\pi i)^k \mathcal{B}_k$$

  Conjectured by Broadhurst and Roberts. Proved and extended by Fresán, Sabbah and Yu

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- Known as Kloosterman motive, framework: Irregular Hodge theory
- $=>$ Oscillatory integrals for Dwork superpotentials and corresponding Thom-Sebastiani sums

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- P. Vanhove formula

$$\int_0^\infty I_0(z)K_0(z)^{l+1}\, z\, dz = \frac{1}{2^l}\int_{X_i\geq 0} \frac{1}{1-W(X)}\prod_{k=1}^{l}\frac{dX_k}{X_k}, \quad W=\left(1+\sum_{i=1}^{l+1}X_i\right)\left(1+\sum_{i=1}^{l+1}X_i^{-1}\right)$$

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- Studied as one parametric family

$$(X_0 + X_1 + \ldots + X_m) \left(\frac{x_0}{X_0} + \frac{x_1}{X_1} + \ldots + \frac{x_m}{X_m} \right) = 1/t.$$

Appeared in banana graph Feynman integrals (Bloch, Kerr, and Vanhove; Klemm, Dühr). Calabi-Yau case $m = 5$ (Hulek and Verrill; Candelas c.s.)

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- Reminds us of Givental's MS which connects

$$\text{Oscillatory Integrals} \quad \leftrightarrow \quad \text{Periods}$$

Bessel moments

- P. Vanhove formula

$$\int_0^\infty I_0(z)K_0(z)^{l+1}\, z\, dz = \frac{1}{2^l}\int_{X_i\geq 0}\frac{1}{1-W(X)}\prod_{k=1}^{l}\frac{dX_k}{X_k},\quad W = \left(1+\sum_{i=1}^{l+1}X_i\right)\left(1+\sum_{i=1}^{l+1}X_i^{-1}\right)$$

- Same potential as for 2-Bessel multi-product restricted to diagonal, BUT different integration path
- Studied as one parametric family

$$(X_0+X_1+\ldots+X_m)\left(\frac{x_0}{X_0}+\frac{x_1}{X_1}+\ldots+\frac{x_m}{X_m}\right) = 1/t.$$

  Appeared in banana graph Feynman integrals (Bloch, Kerr, and Vanhove; Klemm, Dühr). Calabi-Yau case $m = 5$ (Hulek and Verrill; Candelas c.s.)
- W_N gives a universal differential form that connects oscillatory integrals in LHS and periods in RHS
- Works not only for diagonal
- Reminds us of Givental's MS which connects

  Oscillatory Integrals $\leftrightarrow$ Periods
- Should work also for $N > 2$, but need to find a corresponding pairing

$$
\begin{tikzcd}
 & \text{Kloosterman motive} \arrow[d, leftrightarrow, "?"'] \\
QH^{\bullet}(\mathbf{P}^{N-1}) \arrow[ur, "\text{FSY}"] \arrow[r, "\text{Kernel}"] \arrow[d, leftrightarrow, "\text{MS}"] & \begin{array}{c} W_N \\ \text{LG-models} \end{array} \\
\text{Dwork family} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Moments } \mathcal{B}_k \arrow[d, leftrightarrow, "?"'] \\
\text{Quantum DE} \arrow[ur, "\text{FSY}"] \arrow[r] \arrow[d, leftrightarrow, "\text{Laplace}"'] & \begin{array}{c} K(x,y,z) \\ \text{Picard-Fuchs} \end{array} \\
\text{Picard-Fuchs} &
\end{tikzcd}
$$

Many thanks for your attention!