Lille 2025

On the arithmetic and modularity of Calabi-Yau manifolds: attractors, periods & counting points over $\mathbb{F}_q$

Xenia de la Ossa
(University of Oxford)

Algebraic geometry, integrable systems and automorphic forms, Lille 26-30 May 2025 geometry of families of Calabi-Yau varieties and that of their moduli spaces

arithmetic of families of Calabi-Yau varieties and modularity properties

PERIODS

physics

- black hole solutions in string theory
- scattering amplitudes

Collaborators

- P. Candelas & F Rodríguez-Villegas early 2000
- P. Candelas, D. van Straten & M Elmi 2019
- P. Candelas, D. van Straten 2021
- P. Candelas, J McGovern & P Kuusela 2021-2024
- P. Candelas & P Kuusela 2024
- P. Candelas, D van Straten, N. Gegelia, in progress 2025
- P. Candelas & E Svanberg, in progress 2025.

(very much)

1. Calabi-Yau varieties ← generalities
2. The attractor mechanism
3. The arithmetic of Calabi-Yau varieties
4. Back to the attractor mechanism
5. Periods & counting points of $\mathbb{F}_q$

Philip's talk: modularity of one parameter families of Calabi-Yau varieties

1 CALABI-YAU VARIETIES

Mathematical objects of interest:

$\underbrace{\text{algebraic varieties}}_{\text{set of solutions of}}$ with certain special properties

Calabi-Yau manifolds

$$\boxed{P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}}$$

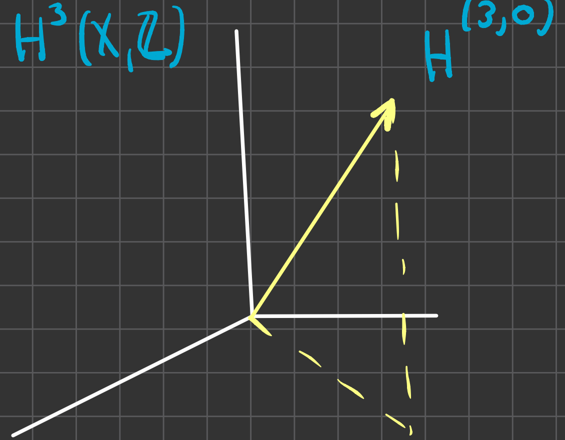
polynomials with complex coefficients φ

CY manifolds: compact Kähler with $c_1 = 0$

admit a Ricci-flat metric

This talk: concerned with $d=3$

- It is a theorem that $\exists !$ (up to a constant) a nowhere vanishing $(3,0)$-form Ω which is holomorphic ($d\Omega = 0$)



$$h^{(3,0)} = \dim H^{(3,0)} = \dim H^{(0,3)} = 1$$

$$\left(H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)} \right)$$

$$\dim H^3 = b_3 = 1 + h^{(2,1)} + h^{(1,2)} + 1 = 2(1 + h^{(2,1)})$$

▸ CY varieties have parameters X_φ

they come in families!

mirror symmetry

complex structure parameters $h^{2,1}$

Kähler structure parameters $h^{1,1}$

(moduli spaces have very $\underbrace{\text{interesting geometry}}_{\text{special geometry}}$)

physics

Examples: very many!

- $\mathbb{P}^4[5]$ eg $\sum_{i=1}^{5} x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 \qquad (x_1, \ldots, x_5) \in \mathbb{P}^4$

  $c_1 = 0$

- We have in mind a particular example:

  Verrill 1996, Hulek & Verrill 2005

  (quotient of the) Mirror of a CICY

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$h^{11} = 5$

quotient

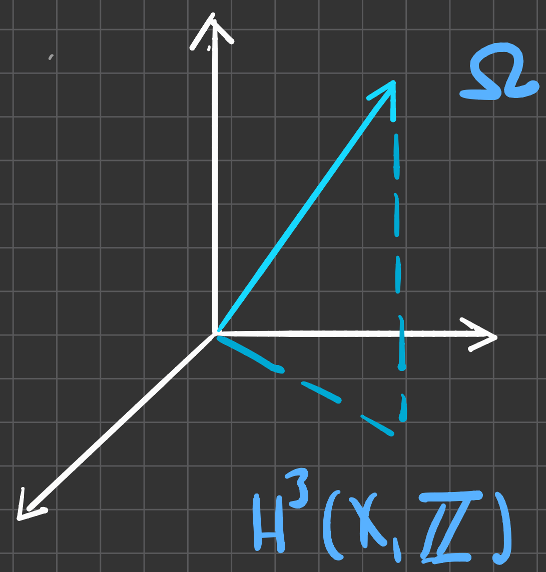
$h^{11} = 1$

why this example? exhibits interesting arithmetic properties which have an interpretation in BH solutions of string theory (also in amplitudes)

Periods and the complex structure

↳ there is a canonical way to give coordinates on the space of complex structures

Recall: Ω is defined up to a scale but is otherwise unique $\Rightarrow$ defines a line in H^3



- study the variations of the complex structure by studying how Ω varies in H^3

- the coordinates of this line are the periods (which then vary as we vary the complex structure)

That is,

$$\Omega(\varphi) = \boxed{z^a(\varphi)}\,\alpha_a - \boxed{\mathcal{F}_a(\varphi)}\,\beta^a$$

where

$\{\alpha^a, \beta_b\} \to$ symplectic basis of $H^3(X,\mathbb{Z})$ $\quad a,b = 0,1,\dots,h^{2,1}$

$$\int_X \alpha_a \wedge \beta^b = -\int_X \beta^b \wedge \alpha_a = \delta_a^b\,, \qquad \int_X \alpha \wedge \alpha = 0\,, \qquad \int_X \beta \wedge \beta = 0$$

let $\{A^a, B_b\}$ the (dual) symplectic basis of $H_3(X,\mathbb{Z})$

Then: $\quad z^a(\varphi) = \int_{A^a} \Omega(\varphi) \qquad \mathcal{F}_a(\varphi) = \int_{B_a} \Omega(\varphi)$

$\{z^a, \mathcal{F}_a\}$ integral basis

Bryant & Griffiths: the complex structure is completely determined by the periods $\{z^a\}$ (so $\mathcal{F}_a = \mathcal{F}_a(z)$). The set $\{z^a(\varphi)\}$ are projective coordinates on the moduli space and we have that $\dim_{\mathbb{C}} \mathcal{M}_{cs} = h^{2,1}$

Periods determine $\underbrace{\text{the geometry of the moduli space}}_{}$

special geometry

- Strominger 91
- Candelas & XD 91

physics!

periods also have arithmetic content!!

Periods are calculable: they satisfy a differential eq of degree b_3 (Picard-Fuchs equation)

To see this (intuitively) consider:

Ω and its variations wrt φ, Ω', Ω'', etc

$\Omega, \Omega', \Omega'', \ldots$ are all closed 3-forms

so at most b_3 of them are linearly independent.

Then there is a linear differential operator $\mathcal{L}$ with $\deg \mathcal{L} = b_3$ st

$$\mathcal{L}\,\Omega = \text{exact 3-form}$$

This implies the periods satisfy the same differential equation as the period integrals are taken over a fixed basis of homology

$$\boxed{\mathcal{L}\, z = 0, \qquad \mathcal{L}\, \mathcal{F} = 0}$$

Focus today
1-parameter examples
$(b_3 = 2(1+h_{21}) = 4)$

This system is known as the Picard-Fuchs equation

- The PF equation is Fuchian, that is the singular points are regular singularities.

Solutions are series around a singularity

$$\sum_{n=-r}^{\infty} a_n \varphi^n$$

series ends from below

and logarithmic solutions are allowed

CY: one such singularity is the MUM point $(T-1)^4=0$, $(T-1)^3 \neq 0$

- There is a prescription to obtain the PF eq

(Dwork and Griffiths; Gelfand, Kapranov and Zelevinskii)

- One parameter families of CY : $b_3 = 4$

$\deg \mathcal{L} = 4$

Solutions around MUM point $\varphi = 0$

$$\varpi_0(\varphi) = \sum_{m=0} a_m \varphi^m = f_0(\varphi)$$

$$\varpi_1(\varphi) = f_0(\varphi)\log\varphi + f_1(\varphi)$$

$$\varpi_2(\varphi) = f_0(\varphi)\log^2\varphi + 2f_1(\varphi)\log(\varphi) + f_2(\varphi)$$

$$\varpi_3(\varphi) = f_0(\varphi)\log^3\varphi + 3f_1(\varphi)\log^2(\varphi) + 3f_1(\varphi)\log\varphi + f_4(\varphi)$$

{Frobenious basis}

These solutions can be found using the method of Frobenious:

solutions of: $$\mathcal{L}\,\varpi(\varphi,\epsilon) = \epsilon^4 \varphi^{\epsilon} \qquad (\epsilon = 0 \Rightarrow \text{PF})$$

of the form $$\varpi(\varphi,\epsilon) = \sum_{m=0}^{\infty} A_m(\epsilon)\,\varphi^{m+\epsilon}$$

with $A_m(0) = a_m$ & $A_0(\epsilon) = 1$

so $\varpi(\varphi, 0) = f_0$

this is a choice

Then $$\varpi_1(\varphi) = \frac{d}{d\epsilon}\varpi(\varphi,\epsilon)\Big|_{\epsilon=0} \qquad \text{etc.}$$

Example: (mirror) quintic

$$P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \,.$$

one parameter family

$$\varphi = (5\psi)^{-5}$$

$$\mathcal{L} = \theta^4 - 5\varphi \prod_{i=1}^{4} (\theta + i/5) \qquad \theta = \varphi \frac{d}{d\varphi}$$

$$A_m(\epsilon) = \underbrace{\frac{\Gamma(5m+5\epsilon+1)}{\Gamma^5(m+\epsilon+1)}}_{\text{at } \epsilon=0 \to a_m} \underbrace{\frac{\Gamma^5(\epsilon+1)}{\Gamma(5\epsilon+1)}}_{\substack{\text{so that} \\ A_0(\epsilon)=1}} \qquad a_m = \frac{(5m)!}{(m!)^5}$$

Verrill : X_φ is $\boxed{\text{smooth}}$ for $\varphi \neq 1, 1/9, 1/25, 0, \infty$

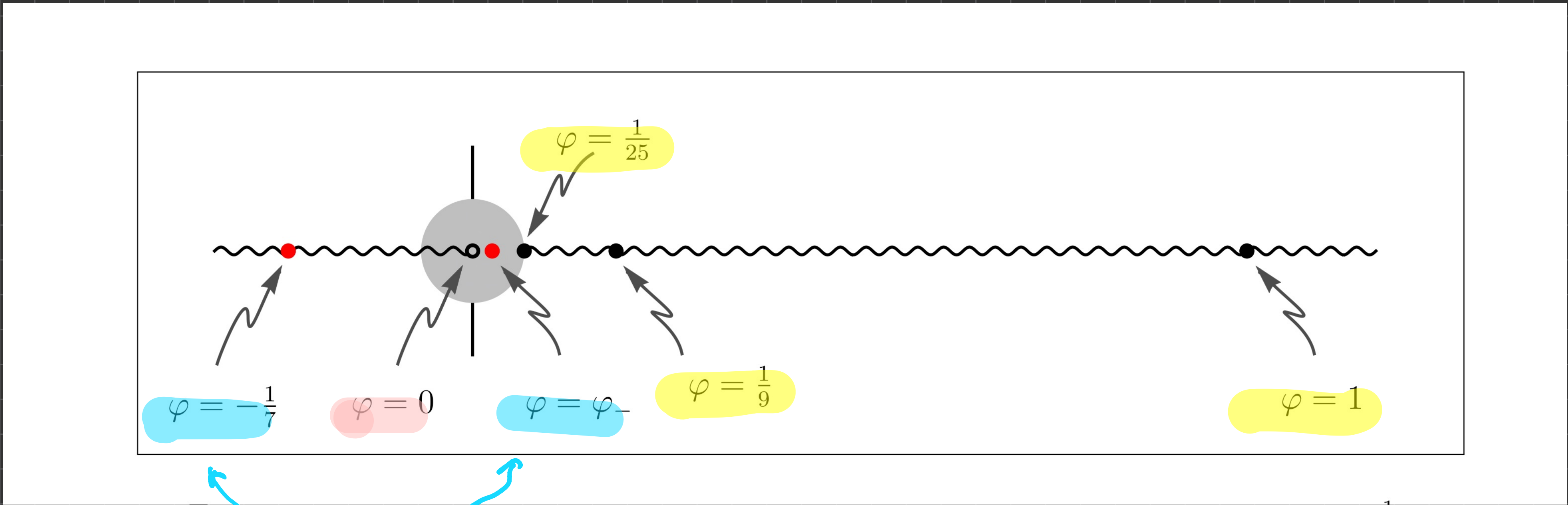
1-parameter case

5 singular cases (not hypergeometric !)

$\varphi = 1, 1/9, 1/25$ conifold type

$\varphi = 0$ LCSL (maximal unipotent monodromy)

$\varphi = \infty$



attractor points (later)

② THE ATTRACTOR MECHANISM

[Ferrara, Kallosh, Strominger 95, ... Greg Moore 98 ....]

Physics : supersymmetric black hole solutions of type IIB superstrings

10 dimensional generalisations of

Einstein's equations for gravity

+

Maxwell equations for electromagnetism

10 dim space-time = $\left(\begin{array}{c} \text{4 dim space-time} \\ \text{spherically symmetric} \\ \text{asymptotically flat} \\ \text{charged BH} \\ \text{parametrised by a} \\ \text{radial coordinate } r \end{array} \right) \times \left(\begin{array}{c} \text{3 } \mathbb{C}\text{dim CY} \\ X_{\varphi(r)} \\ \text{at each point} \\ \text{of the BH} \end{array} \right)$

metric of a 4 dim spherically symmetric asymptotically flat BH

$$ds^2 = -e^{2u(r)}\,dt^2 + e^{-2u(r)}\,d\underline{x}^2, \quad \underline{x} = (x, y, z)$$

(metric dependent only on the radial coordinate r

- asymptotically flat (Minkowski): $u(r) \to 0, \quad r \to \infty$

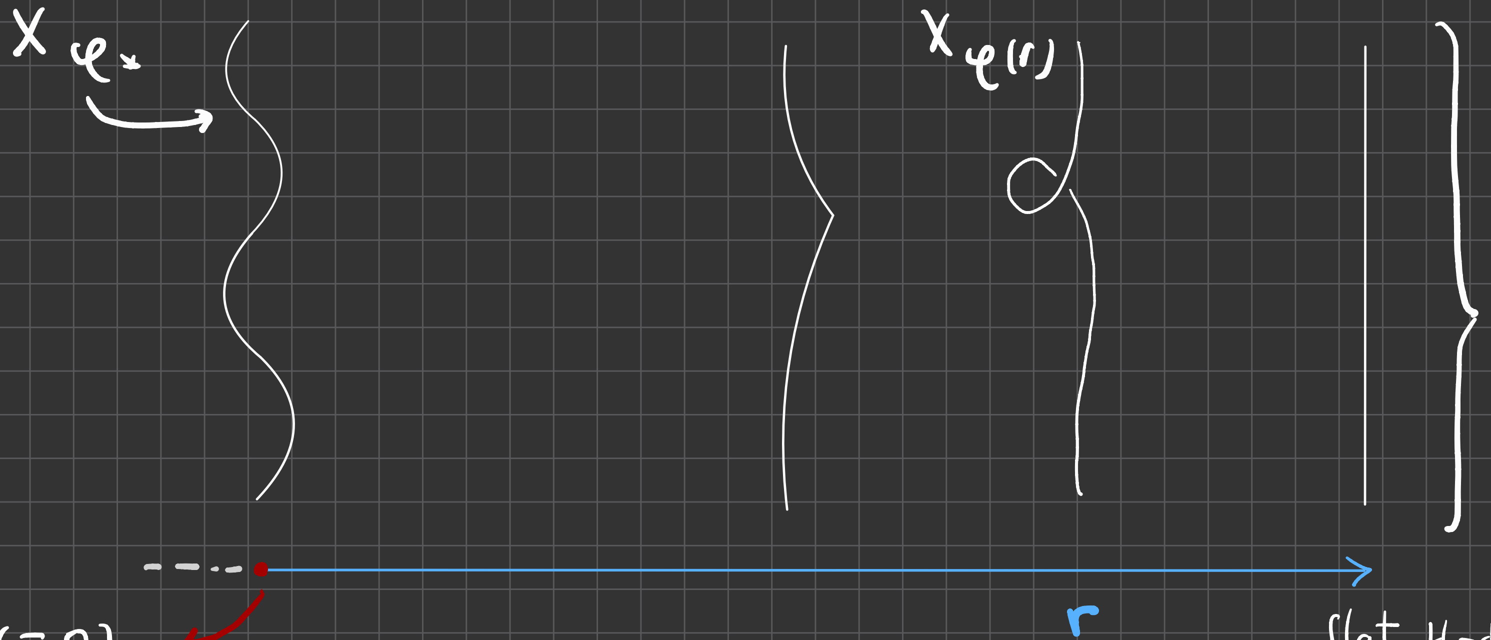
coordinate singularity

- horizon at $r = 0$: $u(r) \longrightarrow -\infty$

last surface visible from infinity (surface separating interior & exterior of the BH)

10 dim space-time : a CY, $X_{\varphi(r)}$, at each point of 4 dim space-time

4 dim space-time = spherically symmetric asymptotically flat charged BH



CY with complex structure φ varying with r

horizon ($r=0$)
(coordinate singularity)

- last surface visible from infinity

flat 4-dim space-time
$r \to \infty$ ($u(r) \to 0$)

$$ds^2 = e^{2u(r)}\,dt^2 + e^{-2u(r)}\,d\underline{x}^2, \quad \underline{x} = (x, y, z)$$

metric dependent only on the radial coordinate r

Type IIB supergravity is gravity with extra $U(1)$ gauge fields (b_3 of them)

So: the BH has electric & magnetic charges

$$Q = \begin{pmatrix} q_a \\ p^b \end{pmatrix} \qquad a, b = 0, 1, \dots, h^{2,1}$$

these must all be integers

Let $\Gamma = p^a \alpha_a - q_a \beta^a \in H^3(X, \mathbb{Z})$

charge vector

(α_a, β^b) symplectic basis of $H^3(X, \mathbb{Z})$

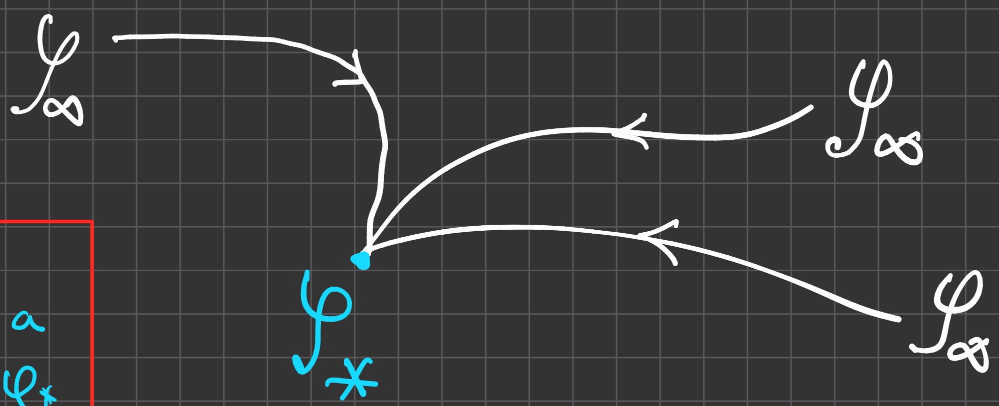
Black hole solutions which preserve supersymmetry need to satisfy <u>1st order differential eqs</u> for $U(r)$ & $\varphi(r)$, <u>the attractor equations</u>

These equations represent a non-linear dynamical system on the $\mathbb{C}$-structure moduli space with flow parameter $\rho = 1/r$

Let $Z_\gamma = e^{K/2} \int_\gamma \Omega$, γ = Poincaré dual of Γ.

Given a choice of $\Gamma \in H^3(X, \mathbb{Z})$ one can prove using the attractor equations that

- the $\mathbb{C}$-structure parameters flow to a value $\varphi_* = \varphi(r=0)$ where $|Z_\gamma|$ reaches a minimum, and it is independent of the starting value $\varphi_\infty = \varphi(r=\infty)$



φ evolves smoothly to a fixed point φ_* at $r=0$

95: Ferrara + Kallosh + Strominger

. . . . .

98: G. Moore conjectures on the arithmetic nature of attractor varieties

$\hookrightarrow X_{\varphi_*} = X_*$

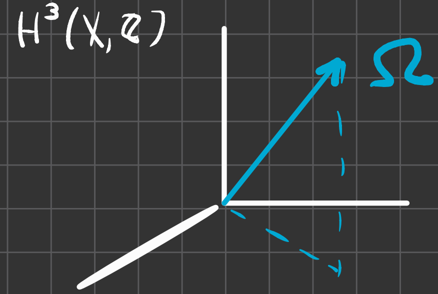
- Moreover,

the Q-structure at an attractor point $\varphi=\varphi_*$ is st

$$\boxed{\begin{gathered}\Gamma = p^a \alpha_a - q_a \beta^a \in H^{(3,0)} \oplus H^{(0,3)} \\ \text{ie} \quad \Gamma^{(1,2)} = \Gamma^{(2,1)} = 0\end{gathered}}$$

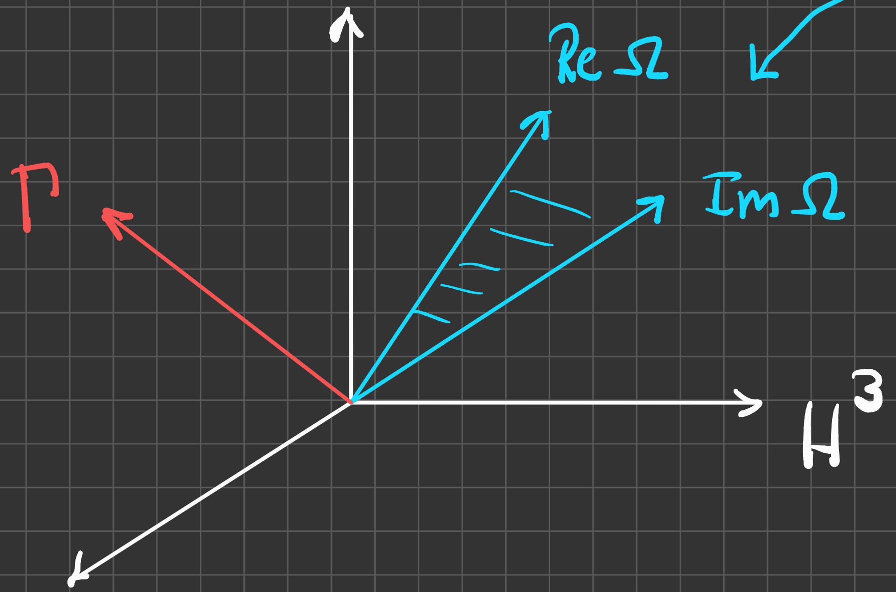
proof: exercise in special geometry

$\boxed{\text{rank 1}}$ attractors



Recall: Ω defines a line in $H^3(X,\mathbb{Z})$

Ω moves inside $H^3(X,\mathbb{Z})$ as we change the CS



Consider

$V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by $\mathrm{Re}\,\Omega$ & $\mathrm{Im}\,\Omega$

$V_{\mathbb{R}}(\varphi)$ moves with φ

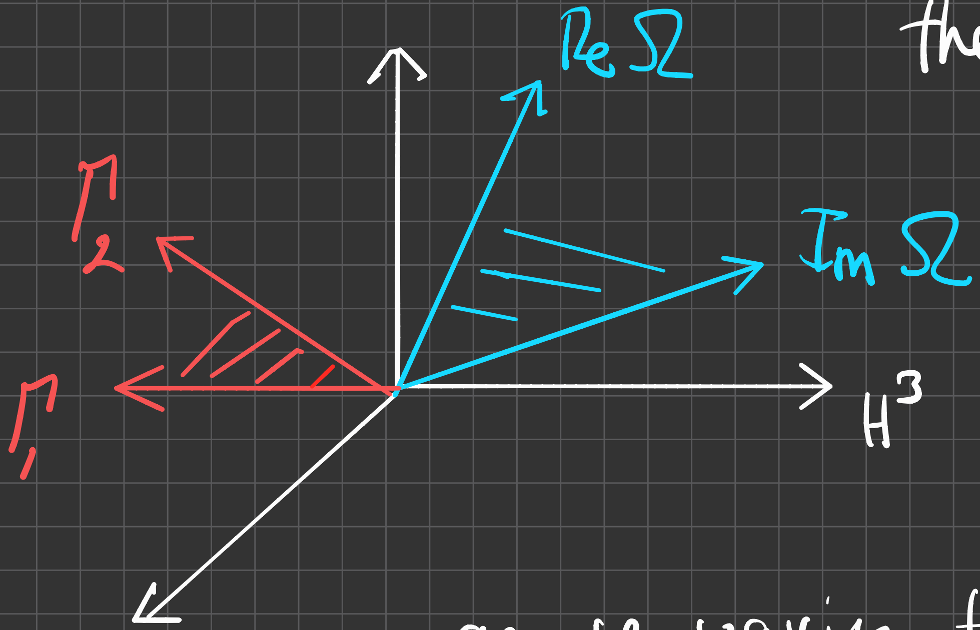
OTOH: Inside $H^3(X,\mathbb{R})$ we have a lattice of vectors $\Gamma \in H^3(X,\mathbb{Z})$ which are fixed

rank 1 A.P: φ st $V_{\mathbb{R}}(\varphi)$ contains the line Γ

$\boxed{\text{rank } 2}$ attractors

at an attractor point of rank 2 there are $\boxed{\text{two}}$ vectors in $H^3(X,\mathbb{Z})$

Γ_1, Γ_2 st $\Gamma_{1,2} \in H^{(3,0)} \oplus H^{(0,3)}$

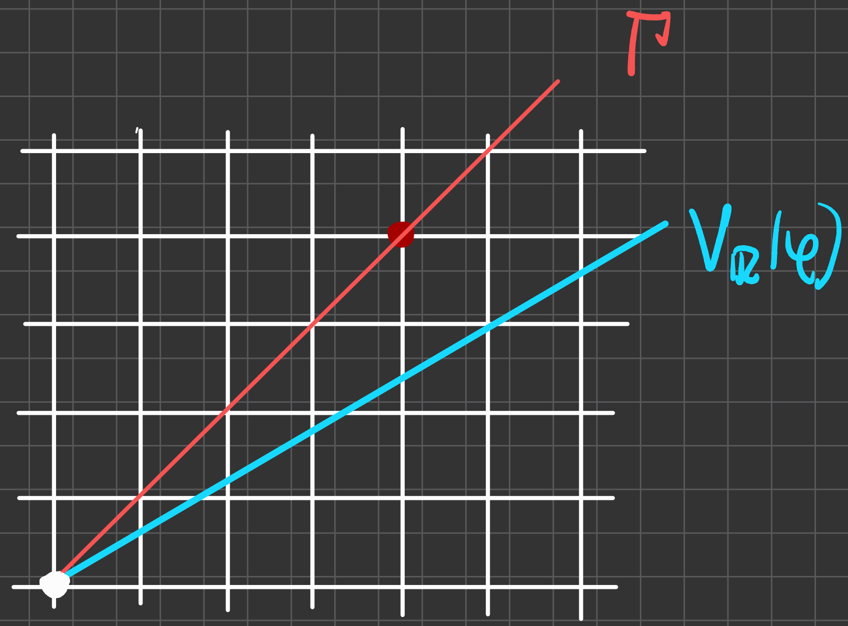


as φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a rank 2 attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ coincides with the plane generated by Γ_1 & Γ_2

RARE, very difficult to find a CY which has rank 2 attractor points

Why?

A line which passes through the origin in general will <u>not</u> pass through another lattice point unless the slope is rational.



Not too hard to find φ st $V_{\mathbb{R}}(\varphi)$ coincides with Γ

For rank 2 attractors we have a plane and it is then <u>much harder</u> to find φ st it coincides with $V_{\mathbb{R}}(\varphi)$

Geometrically: at $\varphi = \varphi^*$

$\Gamma_1, \Gamma_2 \in \Lambda := V_{\mathbb{R}}(\varphi) \cap H^3(X, \mathbb{Z})$ is a lattice plane

and $\Lambda \otimes \mathbb{C} = H^{(3,0)} \oplus H^{(0,3)}$

Also: $\Lambda^\perp \subset H^3(X, \mathbb{Z})$ lattice orthogonal to Λ

(under the natural symplectic product on 3-forms)

satisfies $\Lambda^\perp \otimes \mathbb{C} = H^{(2,1)} \oplus H^{(1,2)}$

We obtain an isomorphism

$$\boxed{H^3(X, \mathbb{Q}) = \Lambda_{\mathbb{Q}} \oplus \Lambda^\perp_{\mathbb{Q}}}$$

$\Lambda \oplus \Lambda^\perp$ has finite index extends to $\mathbb{Q}$

splitting of the Hodge structure $H^3(X, \mathbb{Q})$!

Hodge conjecture $\Rightarrow$ splitting has a geometrical origin

Problem: how do we find varieties with rank 2 attractor points ?

VERY HARD

open question

However: the splitting becomes apparent in the arithmetic structure of X

$\therefore$ arithmetic strategy !

(some progress
P. Candelas + XD + M Elmi + D van Straten
K Bönish + A Klemm + Scheidegger + Zagier
P Candelas + XD + J McGovern)

On the arithmetic and modularity of Calabi-Yau manifolds: attractors, periods & counting points over $\mathbb{F}_q$

2nd talk

Xenia de la Ossa
(University of Oxford)

Algebraic geometry, integrable systems and automorphic forms, Lille 26-30 May 2025

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(very much)

1. Calabi-Yau varieties ← generalities
2. The attractor mechanism
3. The arithmetic of Calabi-Yau varieties ← brief
4. Back to the attractor mechanism
5. Periods & counting points of $\mathbb{F}_q$

Philip's talk: modularity of one parameter families

3 $\boxed{\text{Arithmetic of CY varieties}}$

families of

Let X_φ be a family of algebraic varieties such that X_φ is a hypersurface with defining polynomial $P(\varphi, \underline{X})$

Typically we work with varieties over $\mathbb{C}$.

Instead, work over finite fields $\mathbb{F}_{p^k}$ and compute zeta-functions

So let $\varphi \in \mathbb{F}_p$

The fundamental quantities of interest are

$N_k(\varphi)$ = number of solutions of $P(\underline{x},\varphi)=0$ over $\mathbb{F}_{p^k}$

Generating function ⇝ zeta function

$$\zeta_X(T,p;\varphi) = \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k} N_k(\varphi)\, T^k \right\}$$

depends on both p & φ

properties ⇝ Weil conjectures

(proven by Dwork, Deligne, Grothendieck)

In particular

$$\zeta(T) = \frac{R_1(T)\, R_3(T) \cdots R_{2d-1}(T)}{R_0(T)\, R_2(T) \cdots R_{2d}(T)}$$ rational function

$R_i(T)$ polynomials with integer coeffs

$\deg R_i = b_i$

$R_0(T) = 1 - T$, $R_{2d}(T) = (1 - p^d T)$

For a smooth CY 3-fold X

$$\zeta_X(T) = \frac{\cancel{R_1(T)}\, R_3(T)\, \cancel{R_5(T)}}{(1-T)\, R_2(T)\, R_4(T)\, (1-p^3T)}$$

Weyl conjectures: recall

$\deg R_i = b_i$

- CY 3fold: $b_1 = 0$, $b_5 = 0 \Rightarrow R_1 = 1$ & $R_5 = 1$
- $\deg R_3(T) = b_3$
- $\underset{\underset{b_2}{\parallel}}{\deg R_2(T)} = \underset{\underset{b_4}{\parallel}}{\deg R_4(T)} = h^{11}$ (as $h^{(2,0)} = h^{(0,2)} = 0$)

nicest cases $\longrightarrow$ $R_2(T) = (1-pT)^{h^{11}}$, $R_4(T) = (1-p^2T)^{h^{11}}$

(Picard group generated by divisors defined over the ground field $\mathbb{F}_p$)

1-parameter families $h^{2,1} = 1$

$$b_3 = 1+1+1+1 = 4 \quad \text{so} \quad \deg R_3 = 4$$

$$\zeta_X(T) = \frac{R\,(T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$

smooth CY

$$R\,(T) = 1 + a_p T + b_p\, p\, T^2 + a_p p^3 T^3 + p^6 T^4$$
$$= \det(\mathbb{1} - T\, \mathcal{U}(\varphi))$$

$(a_p = -\operatorname{tr}\mathcal{U},\ 2p b_p = \operatorname{tr}\mathcal{U}^2 - (\operatorname{tr}\mathcal{U})^2)$

$\mathcal{U} = \mathrm{Frob}_3^{-1} : H^3(X) \longrightarrow H^3(X)$ map induced by the Frob automorphism on H^3

4x4 matrix constructed from the periods of Ω

$(a_p = N_p - (1 + h_{11}p + h_{11}p^2 + p^3)$ $\quad$ b_p in terms of N_p & N_{p^2})

singular CY: $\deg R(T) < 4$ eg $\deg R(T) = 3 = 1+2$ conifold

$R(T)$ can be "quickly" computed for $\varphi \in \mathbb{F}_p$ $(\varphi = 0, 1, \ldots, p-1)$ for many primes p

P Candelas, XD & D van Straten (2021) 1 parameter families

P Candelas, XD, P Kuusela (2024) multi parameter families

based on the deformation method of Dwork & Lauder

P Candelas, XD, N Gegelia, D van Straten, in progress

Many questions arise: certainly ready for substancial experimentation.

- One can construct L-functions

$$L(s,\varphi) = \text{"}\prod_p \underbrace{R(p^{-s},\varphi)}_{}{}^{-1}\text{"}$$

spinor zeta function $w=3$

What are the properties of this L-function?

- functional equation
- conductor formula
- modularity $\longleftrightarrow$ paramodular forms of $\Gamma(N) \leq Sp(4)$

Philip's talk

Again at this time these are **hard** questions

modularity is <u>not</u> classical modularity except in some special cases

- rigid CY $(h^{2,1}=0) \Rightarrow \deg R = 2$

$$R(T) = 1 - a_p T + p^3 T^2$$

F. Gouvea + N. Yui: rigid CYs are modular

$a_p \rightsquigarrow$ p-th coeff in q-expansion of a modular form of some $\Gamma_0(N)$, $w=4$

▶ What happens at singularities?

necessary to properly understand the L-function

eg conifold singularities

$$R(T) = (1-\epsilon_p T)(1 - a_p T + p^3 T^2)$$

(for one parameter families)

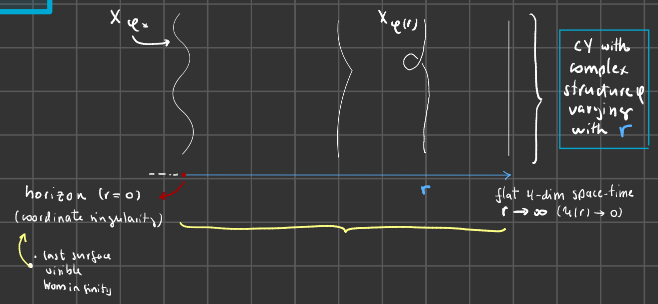
a_p = p-th coeff in q-expansion of the eigenform g of weight 4 of $\Gamma_0(N)$

mirror quintic: $\varphi = 1/5$ $N = 25$ (Schoen)

HV family: $\varphi = 1, 1/9, 1/25$ (not hypergeometric)

↳ next : interesting things happen
for special values of φ
eg attractor points

④ Arithmetic of attractor varieties



Recap: Let X_φ be a CY variety

if φ^* is a rank 2 attractor point in its moduli space

$$\boxed{H^3(X_*, \mathbb{Q}) = \Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}}^{\perp}}$$

$\Lambda \oplus \Lambda^\perp$ has finite index extends to $\mathbb{Q}$

so: Splitting of the Hodge structure $H^3(X, \mathbb{Q})$!

$\Gamma_1, \Gamma_2 \in \Lambda := V_{\mathbb{R}}(\varphi) \cap H^3(X_*, \mathbb{Z})$ is a lattice plane

and $\Lambda \otimes \mathbb{C} = H^{(3,0)} \oplus H^{(0,3)}$

Also: $\Lambda^\perp \subset H^3(X_*, \mathbb{Z})$ lattice orthogonal to Λ

(under the natural symplectic product on 3-forms)

satisfies $\Lambda^\perp \otimes \mathbb{C} = H^{(2,1)} \oplus H^{(1,2)}$

$$\boxed{\text{At } \varphi = \varphi_*}$$

$$H^3(X_*, \mathbb{Q}) = \underbrace{\Lambda}_{(3,0)+(0,3)} \oplus \underbrace{\Lambda^\perp}_{(2,1)+(1,2)}$$

$\mathcal{U} \rightsquigarrow$

$$R_3(T) = \det(\mathbb{1} - T\,\mathcal{U})$$

$$= (1 - p\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

$H^{2,1} \oplus H^{1,2}$ $\qquad$ $H^{3,0} \oplus H^{0,3}$

$\boxed{\text{factors over } \mathbb{Z} \ \forall p}$

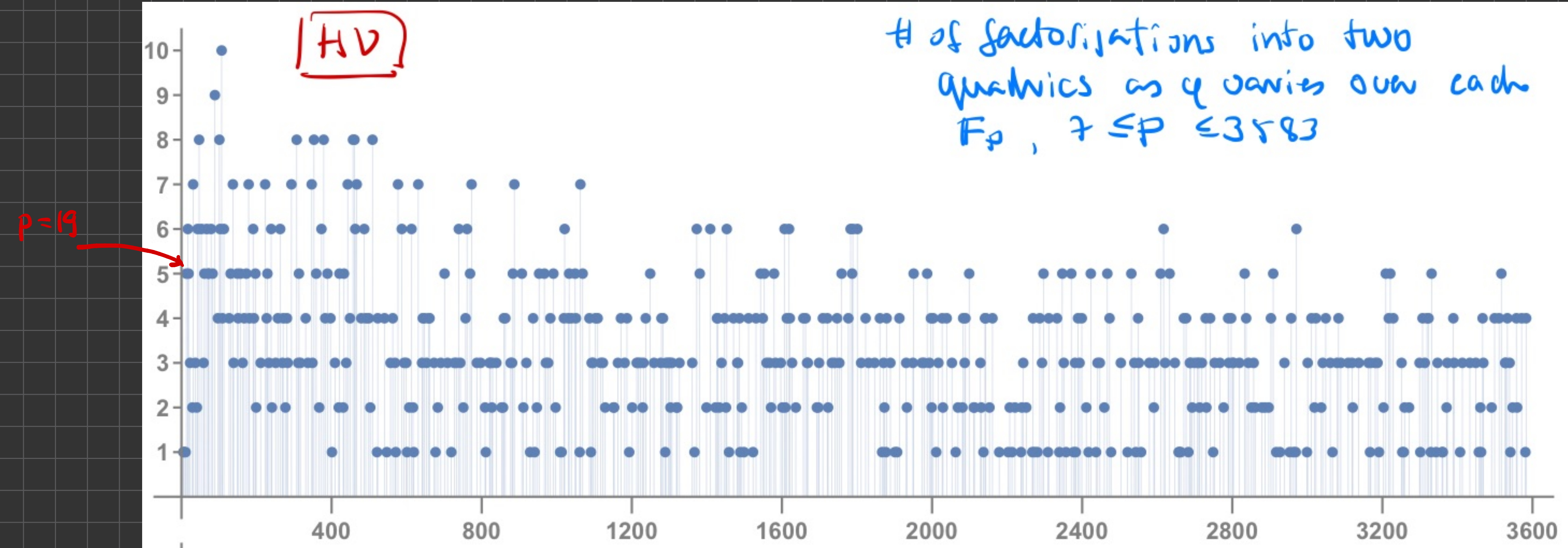
Moreover: expect α_p, β_p to be coeffs of modular forms
(Tate & Serre conjectures)

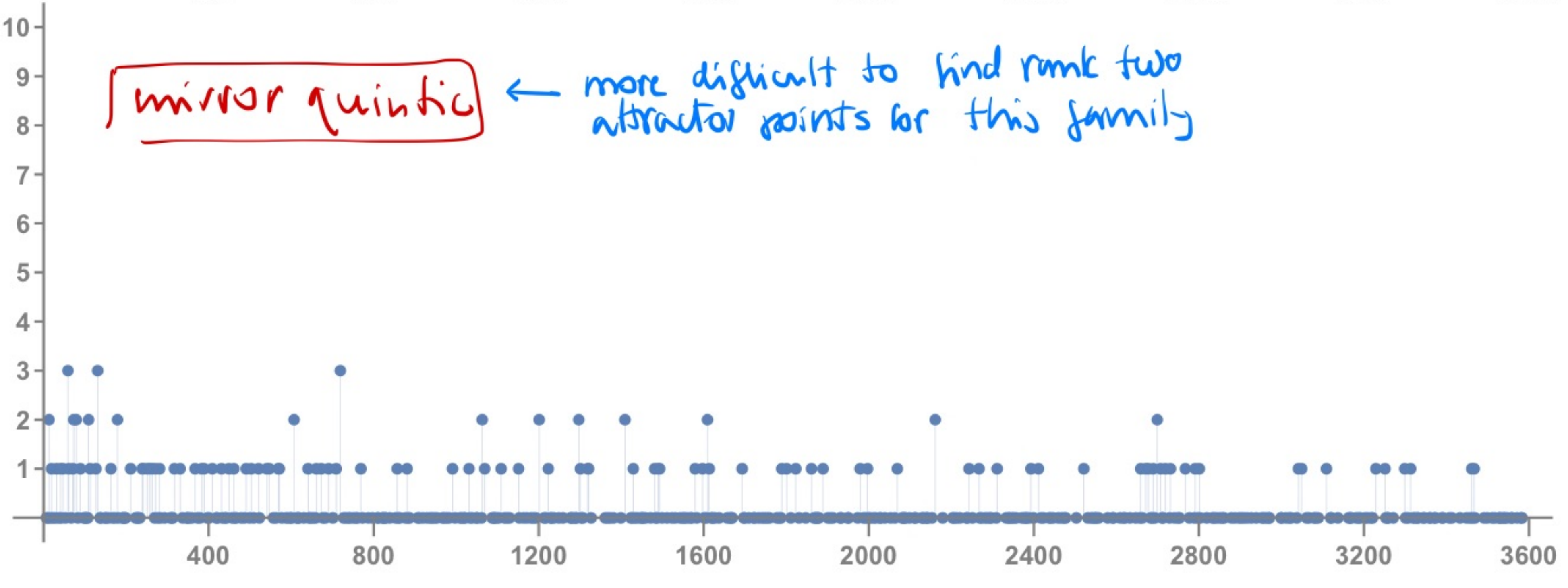
Arithmetic strategy:

make tables of $\mathcal{R}(T,\varphi)$ for <u>many</u> p & φ
and look for persistent factorisations of $\mathcal{R}$
into two quadrics

factorisations occuring when
φ_* is a root of a polynomial
$G(\varphi)$ with integer coeffs

[P. Candelas, XD, A Thorne, D v Straten
P. Candelas, XD, D v Straten 04/2021]


HV
# of factorisations into two quadrics as q varies over each $\mathbb{F}_p$, $7 \le p \le 3583$
p = 19
10
9
8
7
6
5
4
3
2
1
400
800
1200
1600
2000
2400
2800
3200
3600


mirror quintic
more difficult to find rank two attractor points for this family
10
9
8
7
6
5
4
3
2
1
400
800
1200
1600
2000
2400
2800
3200
3600

We find that for the HV manifold there is always a factorisation when (P Candelas, XD, M Elmi & D van Straten)

- $7\varphi + 1 = 0$: $\varphi = -1/7$ $(\forall p \neq 7)$

and

- $\varphi^2 - 66\varphi + 1 = 0$: $\varphi_\pm = 33 \pm 8\sqrt{17}$

(exists in $\mathbb{F}_p$ when 17 = square mod p)

For $p = 19$ (say)

$$\varphi = -\frac{1}{7} \equiv 8 \,, \qquad \varphi_\pm \equiv 4, 5$$

$(17 \equiv 6^2)$

	$p = 19$			
	φ	smooth/sing.	singularity	$R(T)$
conifold	1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
	2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
	3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
φ_+	4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
φ_-	5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
	6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
	7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
$\varphi = -1/7$	8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
	9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
	10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
	11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
	12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
	13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
	14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
	15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
conifold	16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
conifold	17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
	18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

Table 1: *The R-factors for* $\varphi \in \mathbb{F}_{19}$. *Note the factorisations into two quadrics for the five values* $\varphi = 4, 5, 8, 9, 11$.

There is more information in the tables:

there are modular forms

$$R(T) = (1 - p\,\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

Serre's conjecture (generalizing Taniyama-Weil)

"motives" of length two are modular

algebraically defined part of cohomology

[proof: Dieulefait, Khare & Wintenberger, Kisin]

For $\varphi = -1/7$

α_p & β_p are Fourier coefficients of a modular form for $\Gamma_0(14)$

LMFDB

$f_2 = \sum_n \alpha_n q^n$ weight 2 14.2.a.a

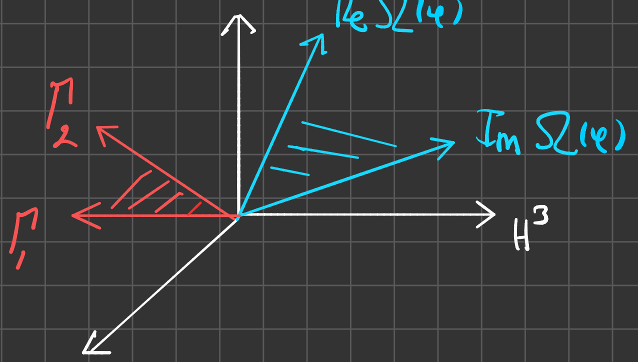
$f_4 = \sum_n \beta_n q^n$ weight 4 14.4.a.a

$q = e^{2\pi i \tau}$ $\tau \in \mathbb{H}$

Associated with these modular forms are L-functions

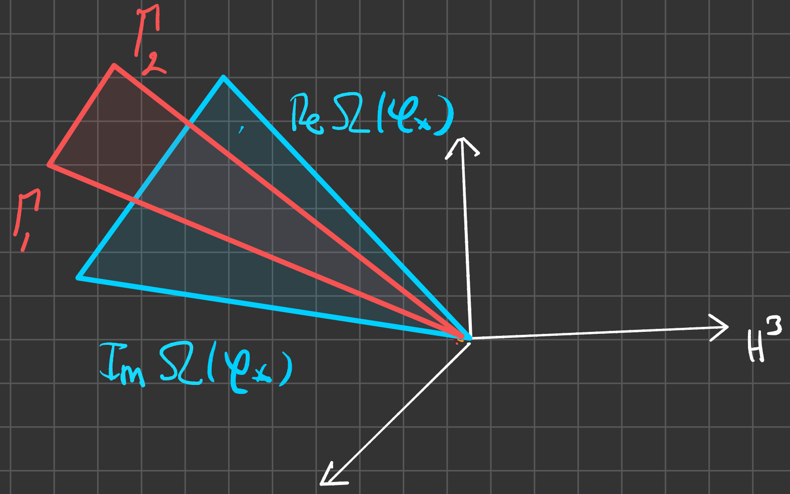
$$L_j(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy \; y^{s-1} f_j(iy) \qquad q = e^{-2\pi y} \qquad j = 2,4$$

At φ_* the periods of Ω are given by simple rational multiples of $L_4(1)$ and $L_4(2)$



$\varphi \to \varphi_*$

$\rightsquigarrow$



For example

$$\Pi\left(-\frac{1}{7}\right) = i\,\frac{L_u(1)}{4\pi}\begin{pmatrix} 8K \\ -30K \\ 0 \\ 5 \end{pmatrix} + \frac{7}{2}\,\frac{L_u(2)}{\pi^2}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

The two integral vectors define a lattice plane

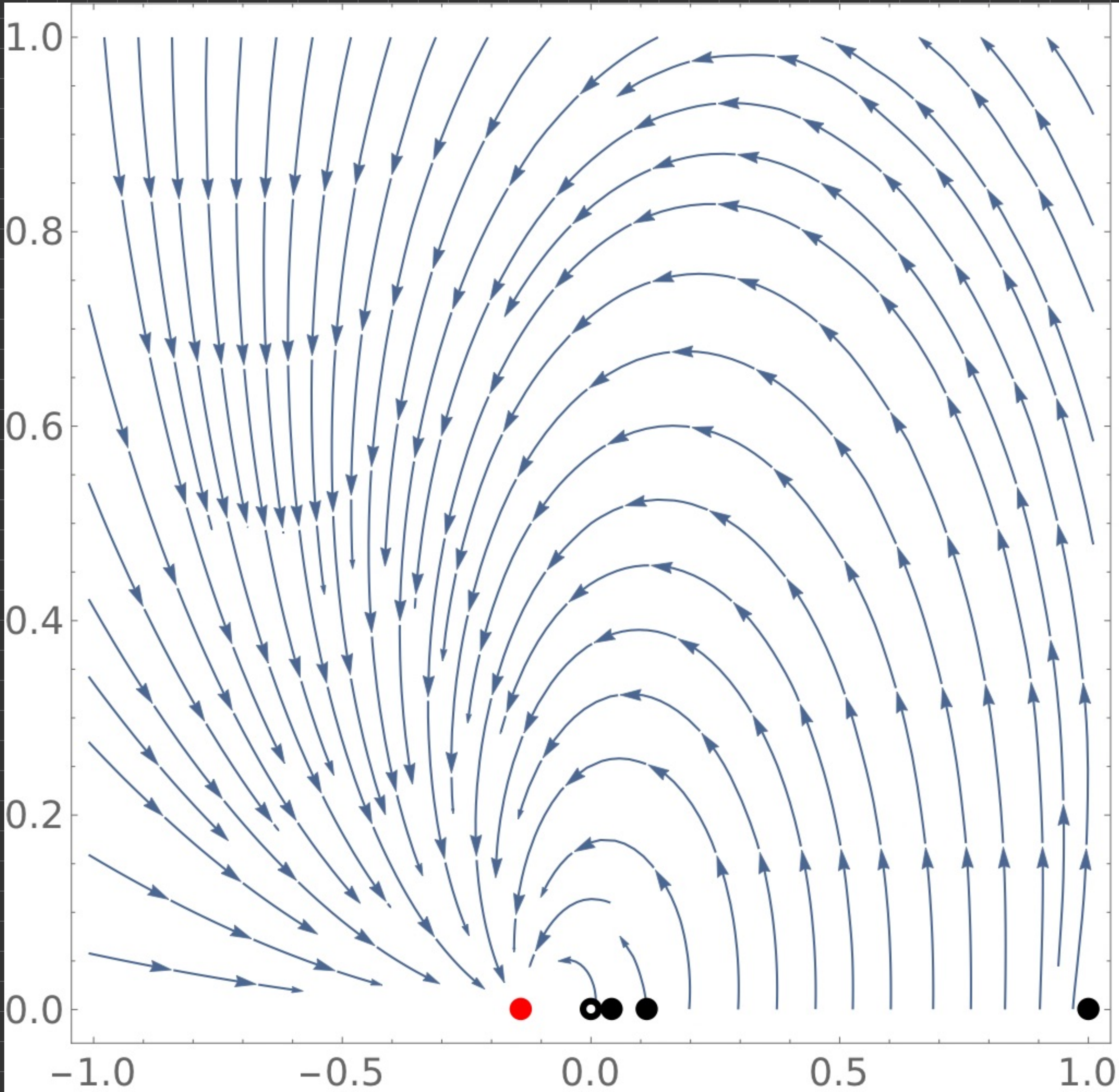
$\Rightarrow$ a two dimensional family of charge vectors.

$$K = \begin{cases} 1 & \text{Verrill}/G_1 \quad h^{21} = 1 \quad h^{11} = 9 \\ 2 & \text{Verrill}/G_1 \times G_2 \quad h^{21} = 1 \quad h^{11} = 5 \end{cases}$$

$$P = \left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} \frac{a_i}{X_i}\right) - \varphi \qquad (X_1, \dots, X_5) \in \mathbb{P}^4, \ \prod X_i \neq 0$$

$$G_1 : X_i \longmapsto X_{i+1} \quad \mathbb{Z}_5 \qquad G_2 : X_i \longmapsto \frac{a_i}{X_i} \quad \mathbb{Z}_2$$

$\varphi = -\frac{1}{7}$



Similarly, for $\varphi_\pm$

α_p & β_p $\longrightarrow$ coeffs of modular forms for $\Gamma_1(34) \subset \Gamma_0(34)$

$SL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod 34$

$f_2 \longrightarrow$ 34.2.b.a

$f_4 \longrightarrow$ 34.4.b.a

Area of the horizon of the BH

$\boxed{\varphi = -1/7}$

Charges $Q_{k\ell} = k(4,-15,-5,0) + \ell(0,0,2,1)$ $\quad k,\ell \in \mathbb{Z}$

(so a two parameter family of BHs)

let $\quad V_* = \dfrac{7}{\pi} \dfrac{L_4(2)}{L_4(1)}$ $\quad$ ($L_4 \to$ L-function associated to f_4)

Then

$$A_{\ell k}(\varphi_*) = 4\pi |Z_\gamma(\varphi_*)|^2 = 14\pi \left\{ k^2 V_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}$$

↳ BH entropy

What is the precise meaning of this ?

Area of the horizon of the BH

$$\boxed{\varphi_\pm = 33 \pm 8\sqrt{17}}$$

φ_+ $$Q_{k\ell} = k(4,-9,7,4) + \ell(4,-30,-30,-5)$$

φ_- $$Q_{k\ell} = k(-2,0,0,5) + \ell(0,3,1,0)$$

$$v_* = \frac{17}{2}\left(\frac{9-\sqrt{17}}{2}\right)\frac{\lambda_4(2)}{\pi\,\lambda_4(1)} \qquad (\lambda_4 = \mathrm{Re}\, L_4)$$

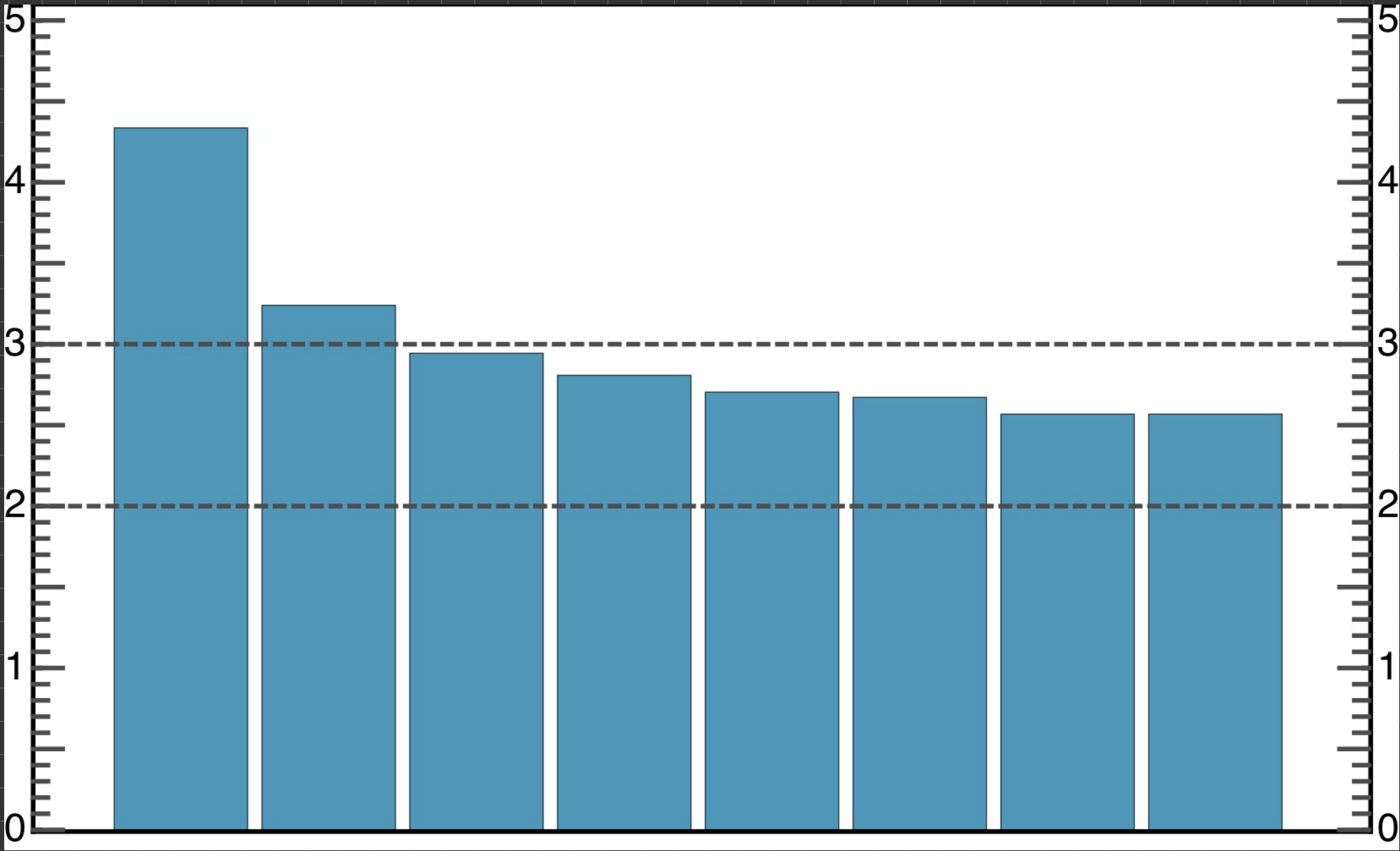
$$A_{k\ell}(\varphi_\pm) = 34\pi\left(\frac{k^2}{v_*} + \ell^2 v_*\right)$$

Corollary to the Chebotariv-theorem

Over $\mathbb{F}_p$, an irreducible polynomial $f(x)$ with integer coefficients has on average one root (over p)

$\Rightarrow$ if $G(x)$ has k irreducible factors it has on average k roots

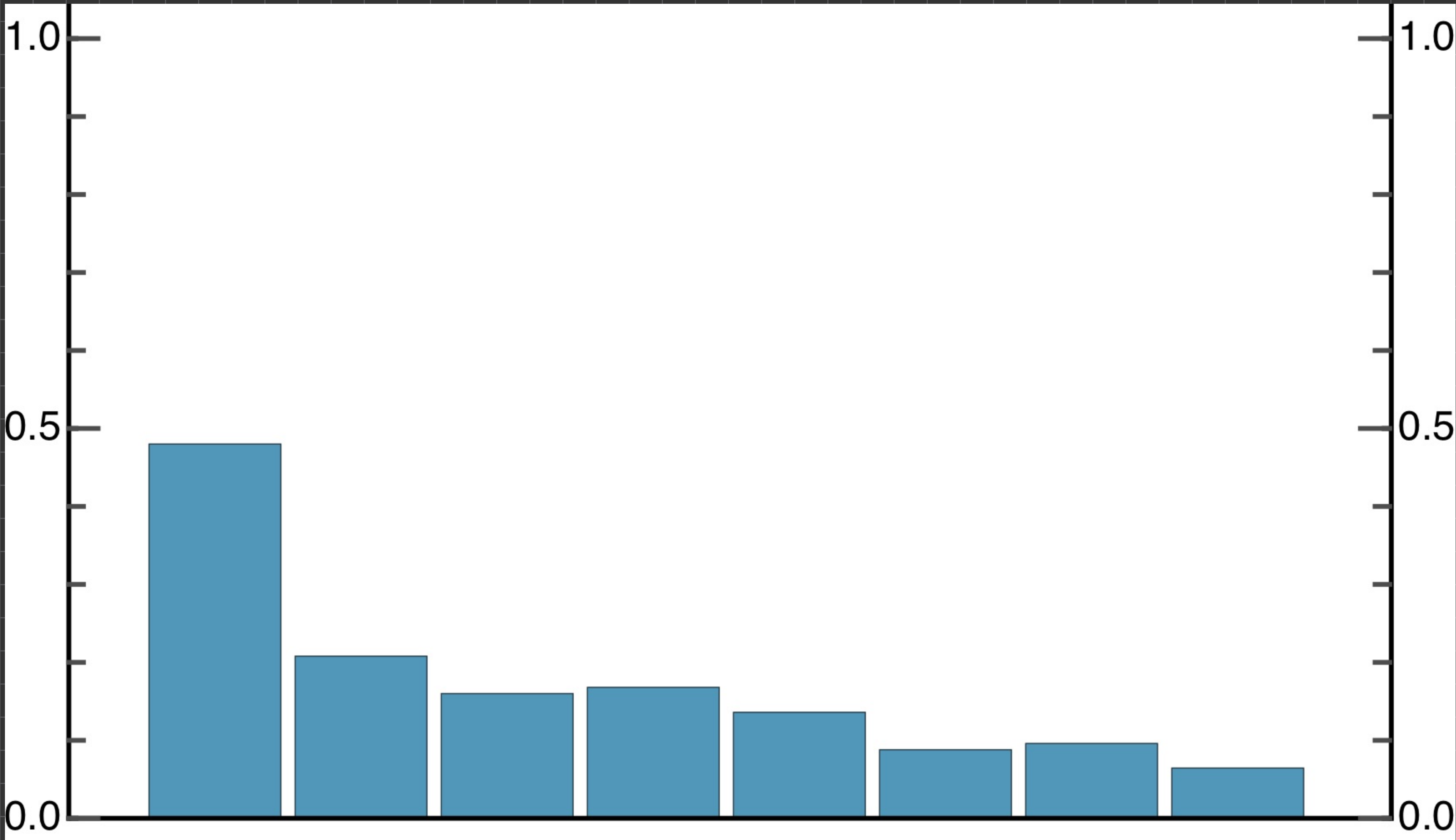
For Verrill's manifold



125

→ 1000 primes

For the mirror quintic



Mirror symmetry

$$\mathrm{QFT}_{IIB}[X] \equiv \mathrm{QFT}_{IIA}[Y]$$

$$\mathcal{M}_{CS}(X) = \mathcal{M}_{KC}(Y)$$

$$\varphi \longmapsto \varphi(t) \qquad \text{mirror map}$$

attractor mechanism

Identities from mirror symmetry

$$\frac{1}{17}(9+\sqrt{17})\frac{\pi\lambda_4(1)}{\lambda_4(2)} = \frac{8}{\sqrt{15}}\left\{1-\sum_{j=1}^{\infty}\sum_{\mathfrak{p}\in\mathrm{pt}(j)} a_{\mathfrak{p}}N_{\mathfrak{p}}\left(\frac{j}{4\pi\sqrt{15}}\right)^{\ell(\mathfrak{p})}\boldsymbol{k}_{\ell(\mathfrak{p})-1}\left(\frac{\sqrt{15}}{3}\pi j\right)\right\}.$$

$N_k = \sum_{d|k} d^3 n_d$

$= k^3 N_k^{GW}$

GW invariants!

Bessel function of the 3rd kind

$k_n(z) = \sqrt{\frac{2}{\pi z}}\, K_{n+\frac{1}{2}}(z)$

modified BF of the 2nd kind

mirror map $\boxed{2\pi i t = \frac{\varpi_1(\varphi)}{\varpi_0(\varphi)}}$

at φ_-: $\quad t(\varphi_-) = i\,\frac{5}{16\cdot 17}(9+\sqrt{17})\,\frac{\pi\lambda_4(1)}{\lambda_4(2)} = \frac{i5}{32}\,\frac{1}{V_*}$

Candelas, McGovern, Kuusela 2021; Candelas, XD, McGovern 2024

Outlook

- What makes a CY an attractor variety?

  rigorous proof: find the geometric reason for the splitting of the Hodge structure

- Mirror symmetry
  - L function of a manifold and its mirror
    (• PC, XD & FRV, 2004 • Daqing Wan, 2006
    • C Doran, T. Kelly, A Salerno, S Sperber, J. Voight, U. Whitcher, 2018 x2
    • A Salerno + U Whitcher 2021 , . . .)
  - Identities

    GW invariants $\longleftrightarrow$ L-values & point counting

(P Candelas, P Kuusela, J McGovern (2021) ; P Candelas, XD, J McGovern (2024))

• . . .

▶ Modularity of CY varieties ?

1-parameter families $\rightsquigarrow$ paramodular forms of $\Gamma(N) \subseteq Sp(4)$

$\rightsquigarrow$ at certain values of the parameter this "degenerates" into known modular forms of $\Gamma_0(N) \subseteq SL(2,\mathbb{Z})$

(e.g. attractor varieties, conifolds)

$\rightsquigarrow$ Examples from IIB [X] flux vacua

Conjecture: (Kachru, Nally, Yang) A CY variety X_φ with a.s. φ which gives a supersymmetric flux vaccum is modular in the sense that it is associated to a modular form of $w=2$

(. . . . . Candelas, XD, McGovern, Kuusela 2023)

THANKS!