Affine Weyl groups and non-abelian discrete systems

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Motivation & Outline

Motivation

- Painlevé equations being one of the most important objects in mathematics and mathematical physics have various types of non-abelian analogs: quantum [Nagoya, 2004], matrix differential [Kawakami, 2015], matrix difference [Cassatella-Contra et al., 2014], non-abelian differential [Bobrova and Sokolov, 2023b].
- Some of them are connected with integrable non-abelian PDEs [Olver and Sokolov, 1998] and PΔEs [Adler, 2020], Riemann-Hilbert problem [Cafasso and Manuel, 2014], orthogonal polynomials [Cafasso et al., 2018], Calogero systems [Bertola et al., 2018], and etc.
- In the commutative case, discrete Painlevé equations have been studied in a series of papers by B. Gramaticos and A. Ramani since 1990s, but without understanding the whole picture.
- The latter was clarified by H. Sakai in his famous paper [Sakai, 2001].
- We would like to derive the same picture in the non-commutative case.
- But we first present an algebraic tool in order to obtain good examples for the further study.
- It uses the affine Weyl groups and might be regarded as a non-abelian analog of that suggested in [Noumi and Yamada, 1998].

Outline

1. A brief introduction to the Painlevé equations and their non-abelian analogs.
2. Affine Weyl groups and discrete dynamics: commutative and non-commutative cases.
3. Non-abelian dressing chain and related discrete systems.
4. Non-abelian difference discrete Painlevé equations.
5. Further questions.

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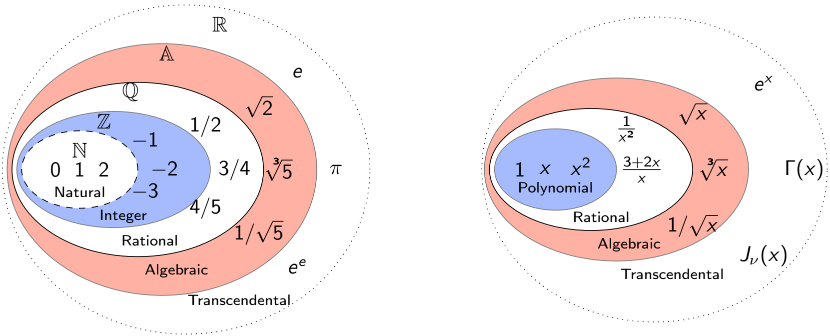

Number systems and function classes [Joshi, 2020]

- Problem: define new functions by an ODE of the m^{th} order with properties that generalize those of elliptic functions. [L. Fuchs], [H. Poincaré]
- Painlevé property: the general solution of an ODE has no critical movable points.
- $m = 2$: six classes defining the Painlevé transcendents. [Painlevé, 1902], [Gambier, 1910]
- P_1 transcendent: $y''(z) = 6y(z)^2 + z$. (a non-autonomous analog for the $\wp$-function!)

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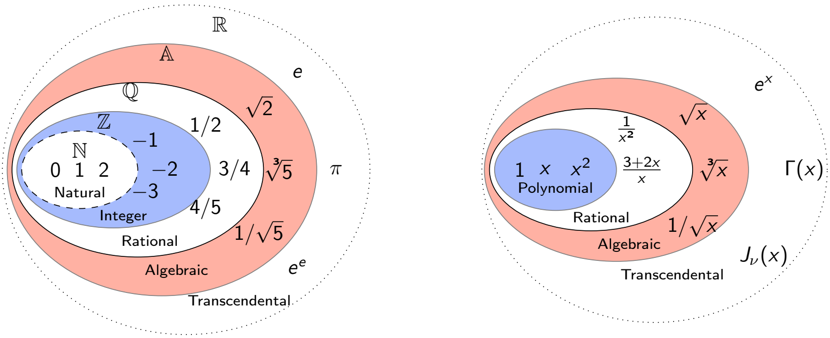

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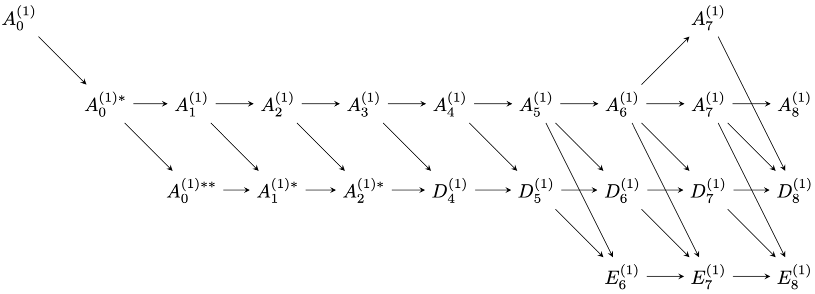

Sakai's classification scheme: the surface type [Sakai, 2001]

(picture source: [Dzhamay and Takenawa, 2018])

- Problem: classify discrete Painlevé equations by using a compact rational surface X with a unique canonical divisor D of canonical type.
- Main idea: to each X, there corresponds a root subsystem of $E_8^{(1)}$ inside a Picard lattice of X.
- Classification: 22 discrete systems of elliptic, multiplicative, or additive types.
- d-$P_1(E_7)$: $T\,(q,p,t;\alpha_0,\alpha_1) = \left(-q-\alpha_1 p^{-1},\ -p+2\bar{q}^2+t,\ t;\ \alpha_0-1,\ \alpha_1+1\right) \overset{\text{cont.lim.}}{\longrightarrow} \mathrm{P}_1$

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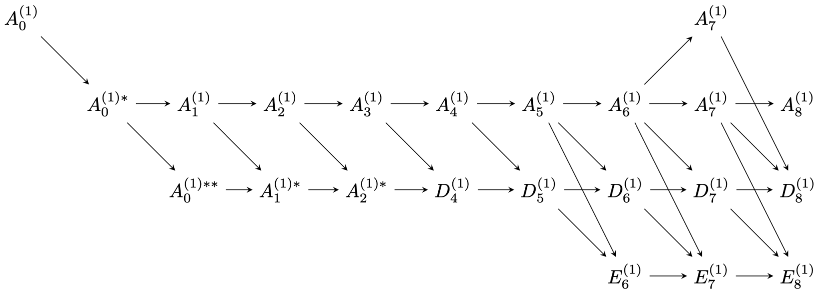

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Why non-abelian?

The Ablowitz-Ramani-Segur conjecture [Ablowitz et al., 1980]

> A nonlinear PDE is solvable by the inverse scattering method [Zakharov and Shabat, 1974] only if every nonlinear ODE obtained by an exact reduction has the Painlevé property.

- Many important integrable PDEs and PΔEs can be solved in terms of the Painlevé equations.

Examples.

- KdV $\to$ P_1, P_2
- sin-Gordon $\to$ P_3
- NLS $\to$ P_4
- VL $\to$ dP_1

- These integrable systems have been intensively studied in the non-commutative setting.
- Their analogs have similar integrability property generalized to the non-commutative case.
- So, it is natural to investigate their "solutions".

A matrix KdV equation [Wadati and Kamijo, 1974]

$$w_t + 6\,ww_x + 6\,w_x w + w_{xxx} = 0, \qquad w = w(x,t) \in \mathrm{Mat}_n(\mathbb{C}), \qquad x,\ t \in \mathbb{C}. \qquad \mathrm{KdV}^0$$

- The inverse scattering method [Wadati and Kamijo, 1974].
- A hierarchy of commuting symmetries [Olver and Sokolov, 1998], [Olver and Wang, 2000].
- The ZCR $\quad \partial_t U - \partial_x V = [V, U], \quad$ where $U = U(\mu, x, t)$ and $V = V(\mu, x, t)$.
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From affine Weyl groups to discrete dynamics

Affine Weyl groups

- Let us fix a generalized Cartan matrix $C = (c_{ij})$, where $i, j \in I := \{0, 1, \dots, n\}$.
- Sets $\Delta = \{\alpha_0, \dots, \alpha_n\}$, $\Delta^\vee = \{\alpha_0^\vee, \dots, \alpha_n^\vee\}$ correspond to simple roots and simple co-roots.
- Denote by $Q = Q(C)$ and $Q^\vee = Q^\vee(C)$ the root and co-root lattices.
  The pairing $\langle \cdot, \cdot \rangle : Q \times Q^\vee \to \mathbb{Z}$ is defined by $\langle \alpha_i, \alpha_j^\vee \rangle = c_{ij}$ and $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$.
- Denote by $W = W(C)$ the Weyl group (or the Coxeter group) defined by generators s_i, $i \in I$:

$$W(C) = \langle s_0, s_1, \dots, s_n \,|\, s_i^2 = 1,\ (s_i\, s_j)^{m_{ij}} = 1 \rangle, \tag{1}$$

  where the exponents are determined by the value of the product $c_{ij} c_{ji}$ as below

$c_{ij} c_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

- These generators act naturally on Q by reflections

$$s_i(\alpha_j) = \alpha_j - \langle \alpha_i, \alpha_j^\vee \rangle\, \alpha_i = \alpha_j - c_{ij}\, \alpha_i. \tag{2}$$

- Each s_i-action on Q induces an automorphism of the $\mathbb{C}(\alpha)$ of rational functions in α_i.
  Hence, $\mathbb{C}(\alpha)$ is a left W-module.
- Recall that one of the important properties of the affine Weyl groups is that they have translations, also known as Kac translations. Let W_0 be a finite Weyl group, $\delta = \sum_{i \in I} k_i\, \alpha_i$ be the null root and $V_0 = \{\mu \in V \,|\, \langle \mu, \delta^\vee \rangle = 0\}$. For an element $\mu \in V_0$ such that $\langle \mu, \mu^\vee \rangle \neq 0$ we define a translation element $t_\mu \in W$ by the formula

$$t_\mu = s_{\delta - \mu}\, s_\mu \tag{3}$$

  and suppose that $w\, t_\mu = t_{w(\mu)}\, w$ for any $w \in W$.

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Extended birational representations [Noumi and Yamada, 1998]

- The Kac translation acts on simple affine roots as follows

$$t_\mu(\alpha) = \alpha - \langle \mu, \alpha \rangle \, \delta = \alpha - \mu_\alpha \delta. \tag{4}$$

- It is known that the affine Weyl group is decomposed into a semi-direct product of translations in the lattice part M and the finite Weyl group W_0 acting on M, i.e. $W = M \rtimes W_0$. The lattice part M acts on $\mathbb{C}(\alpha)$ as a shift operator, thanks to (4).
- Let us consider the set of functions f_i, $i \in I$, which we will often call variables.
- We propose an extension of the representation of W on $\mathbb{C}(\alpha)$ to the field $\mathbb{C}(\alpha, f)$ of rational functions in α_i and f_i, $i \in I$. One needs to specify the action of s_i on f_j in such a way that the automorphisms s_i on $\mathbb{C}(\alpha, f)$ preserve the Weyl group structure.

Remark 1. Such classes of representations arise naturally from Bäcklund transformations of the differential Painlevé equations.

Remark 2. Sometimes it is necessary to work with an extended Weyl group $\widetilde{W}$, which is a semi-direct product of W and the group Ω of automorphisms of the Dynkin diagram $\Gamma(C)$, i.e.

$$\widetilde{W} = W \rtimes \Omega.$$

An automorphism of $\Gamma(C)$ is a bijection π on I s.t. $c_{\pi(i)\pi(j)} = c_{ij}$ and, therefore, $\pi \, s_i = s_{\pi(i)} \, \pi$. Note that the representations of W lifts to a representation of $\widetilde{W}$.

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Discrete dynamics [Noumi and Yamada, 1998]

- Suppose that we extended the action of W from $\mathbb{C}(\alpha)$ to $\mathbb{C}(\alpha, f)$. Here we consider an arbitrary extension $\mathbb{C}(\alpha, f)$ as a left W-module, assuming that each element of W acts on the function field as an automorphism.
- For each $\mu \in M$ we define a set of rational functions $F_{\mu,i}(\alpha, f) \in \mathbb{C}(\alpha, f)$ by

$$t_\mu(f_i) = F_{\mu,i}(\alpha, f). \tag{5}$$

- This set can be considered as a discrete dynamical system.
- Here α_i and f_i are *discrete time variables* and *depended variables* respectively.
- Based on the action of t_μ on the discrete time variables, discrete dynamics can be classified into additive (*d-equations*), multiplicative (*q-equations*), or elliptic (*ell-equations*) types.

Remark. A similar description of the discrete dynamics can be given for $\widetilde{W}$ as well.

- Now it is clear how to generalise this construction to the non-abelian case.
  We just need to consider "non-commutative" f_i and repeat the same arguments as above.

Non-abelian setting

- Consider an associative unital division ring $\mathcal{R}$ over the field $\mathbb{C}$ equipped with a derivation.
- We assume that all greek letters belong to the field $\mathbb{C}$, while the elements f_i are from $\mathcal{R}$. We will often call f_i as *functions*.
- The derivation $d_t : \mathcal{R} \to \mathcal{R}$ of the ring $\mathcal{R}$ is a $\mathbb{C}$-linear map satisfying the Leibniz rule. We also assume that there is a central element t such that $d_t(t) = 1$ and for any $\alpha \in \mathbb{C}$ we have $d_t(\alpha) = 0$. Here and below we identify the unit of the field with the unit of the ring.
- For the brevity we denote $d_t(f_i) = \dot{f}_i$, $d_t^2(f_i) = \ddot{f}_i$, and so on.
- Note that on $\mathcal{R}$ we have an involution called the transposition τ, which acts trivially on the generators of $\mathcal{R}$ and for any elements $F, G \in \mathcal{R}$ we have $\tau(F\,G) = \tau(G)\,\tau(F)$. This involution can be naturally extended to the matrices over $\mathcal{R}$.

Remark. We would rather not specify the generators of the ring $\mathcal{R}$ in order to avoid overloaded description of a pretty simple thing. Instead, we encourage the reader to think of the ring $\mathcal{R}$ as a generalization of rational functions over the field $\mathbb{C}$ to a non-abelian case.

Non-abelian discrete dynamics [Bobrova, 2024]

- Consider the set of elements $f_i \in \mathcal{R}$, $i \in I$, which we will often call *functions* or *variables*.
- Repeating all the arguments above, we again suppose that the action of W is extended from $\mathbb{C}(\alpha)$ to $\mathbb{C}(\alpha, f)$ and for each $\mu \in M$ we define a set of elements $F_{\mu,i}(\alpha, f) \in \mathbb{C}(\alpha, f)$ by

$$t_\mu(f_i) = F_{\mu,i}(\alpha, f). \tag{6}$$

  This set can be considered as a non-abelian discrete dynamical system.

Remark. Up to the author's knowledge, examples of non-abelian discrete systems of *ell*-type have not appeared yet, while there exist examples of q- and d-types systems. Systems of q-type might be found in [Bobrova et al., 2023], where the non-commutative analogs of the q-P_1 and q-P_2 hierarchies are presented. Examples of non-abelian discrete d-systems can be found, for instance, in [Cassatella-Contra et al., 2014] and [Adler, 2020].

- A suitable birational representation of W leads to a discrete dynamical system.
- Note that classes of such representations arise naturally from Bäcklund transformations of ordinary differential equations, in particular, of the Painlevé equations.

Dynamics related to the dressing chain

Non-abelian dressing chains

Remark. The commutative dressing chain was introduced in [Veselov and Shabat, 1993]. It is related to the Painlevé equations [Adler, 1994] and arises from a generalisation of the symmetries for the P_4 and P_5 systems [Noumi and Yamada, 2000]. Quantum dressing chain might be found in [Nagoya, 2004]. Here we **do not** assume any relations on the elements f_i.

- Let $j \in {}^{\mathbb{Z}}\!/\!_{(n+1)\mathbb{Z}}$. Consider the systems for $n = 2l$ and $n = 2l+1$, $l \in \mathbb{Z}_{\geq 0}$, respectively

$$\dot{f}_j = \sum_{1\leq r\leq l} f_j\, f_{j+2r-1} - \sum_{1\leq r\leq l} f_{j+2r}\, f_j + \alpha_j; \qquad A_{2l}^{(1)}$$

$$\begin{aligned} \tfrac{1}{2}\, t\, \dot{f}_j = & \sum_{1\leq r\leq s\leq l} f_j\, f_{j+2r-1}\, f_{j+2s} - \sum_{1\leq r\leq s\leq l} f_{j+2r}\, f_{j+2s+1}\, f_j \\ & + \Big(\tfrac{1}{2} - \sum_{1\leq r\leq l} \alpha_{j+2r}\Big) f_j + \alpha_j \sum_{1\leq r\leq l} f_{j+2r}. \end{aligned} \qquad A_{2l+1}^{(1)}$$

- We will cal them $A_n^{(1)}$, $n \geq 2$ type systems or *dressing chains in the Noumi-Yamada variables*.
- These systems admit Lax pairs.

Lax pairs

- Let $\Psi = \Psi(\lambda, t) \in \mathrm{Mat}_{n+1}(\mathcal{R})$, $\lambda \in \mathcal{Z}(\mathcal{R})$ satisfy the linear system

$$\begin{cases} \partial_\lambda \Psi(\lambda, t) & = & \mathcal{A}(\lambda, t)\, \Psi(\lambda, t), \\ \partial_t \Psi(\lambda, t) & = & \mathcal{B}(\lambda, t)\, \Psi(\lambda, t), \end{cases} \tag{7}$$

where matrices $\mathcal{A} = \mathcal{A}(\lambda, t)$ and $\mathcal{B} = \mathcal{B}(\lambda, t)$ belong to $\mathrm{Mat}_{n+1}(\mathcal{R})$ and are of the form

$$\mathcal{A}(\lambda) = A_0 + A_{-1}\, \lambda^{-1}, \qquad \mathcal{B}(\lambda) = B_1\, \lambda + B_0. \tag{8}$$

- Consider the matrices expressed in terms of the standard unit matrices $E_{r,s} \in \mathrm{Mat}_{n+1}(\mathbb{C})$ as

$$A_0 = E_{1,n} + f_0\, E_{1,n+1} + E_{2,n+1}, \quad A_{-1} = \sum_{1 \le r \le n+1} \beta_r\, E_{r,r} + \sum_{1 \le r \le n} f_r\, E_{r+1,r} + \sum_{1 \le r \le n-1} E_{r+2,r},$$

$$B_1 = E_{1,n+1}, \quad B_0 = \sum_{1 \le r \le n+1} g_r\, E_{r,r} + \sum_{1 \le r \le n} E_{r+1,r}.$$

- Let $\alpha_0 = 1 + \beta_{n+1} - \beta_1, \quad \alpha_j = \beta_j - \beta_{j+1}, \quad j \in \mathbb{Z}/(n+1)\mathbb{Z} \setminus \{0\}$.

Theorem. [Bobrova, 2024] There exists a set of the g-functions such that the compatibility condition of system (7) is equivalent to either the $A_{2l}^{(1)}$ or $A_{2l+1}^{(1)}$ system.

- For the $A_{2l}^{(1)}$, we have $g_j = -\sum_{1 \le r \le l} f_{j+2r}$, where indexes belong to $\mathbb{Z}/(n+1)\mathbb{Z}$.

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- Let $\Psi = \Psi(\lambda, t) \in \mathrm{Mat}_{n+1}(\mathcal{R})$, $\lambda \in \mathcal{Z}(\mathcal{R})$ satisfy the linear system

$$\begin{cases} \partial_\lambda \Psi(\lambda, t) & = & \mathcal{A}(\lambda, t)\,\Psi(\lambda, t), \\ \partial_t \Psi(\lambda, t) & = & \mathcal{B}(\lambda, t)\,\Psi(\lambda, t), \end{cases} \tag{7}$$

where matrices $\mathcal{A} = \mathcal{A}(\lambda, t)$ and $\mathcal{B} = \mathcal{B}(\lambda, t)$ belong to $\mathrm{Mat}_{n+1}(\mathcal{R})$ and are of the form

$$\mathcal{A}(\lambda) = A_0 + A_{-1}\,\lambda^{-1}, \qquad \mathcal{B}(\lambda) = B_1\,\lambda + B_0. \tag{8}$$

- Consider the matrices expressed in terms of the standard unit matrices $E_{r,s} \in \mathrm{Mat}_{n+1}(\mathbb{C})$ as

$$A_0 = E_{1,n} + f_0\,E_{1,n+1} + E_{2,n+1}, \quad A_{-1} = \sum_{1\le r\le n+1} \beta_r\,E_{r,r} + \sum_{1\le r\le n} f_r\,E_{r+1,r} + \sum_{1\le r\le n-1} E_{r+2,r},$$

$$B_1 = E_{1,n+1}, \quad B_0 = \sum_{1\le r\le n+1} g_r\,E_{r,r} + \sum_{1\le r\le n} E_{r+1,r}.$$

- Let $\alpha_0 = 1 + \beta_{n+1} - \beta_1, \quad \alpha_j = \beta_j - \beta_{j+1}, \quad j \in {}^{\mathbb{Z}}\!/_{(n+1)\mathbb{Z}} \setminus \{0\}.$

Theorem. [Bobrova, 2024] There exists a set of the g-functions such that the compatibility condition of system (7) is equivalent to either the $A_{2l}^{(1)}$ or $A_{2l+1}^{(1)}$ system.

- For the $A_{2l}^{(1)}$, we have $g_j = -\sum_{1\le r\le l} f_{j+2r}$, where indexes belong to ${}^{\mathbb{Z}}\!/_{(n+1)\mathbb{Z}}$.

Bäcklund transformations and discrete dynamics

- Let the Cartan matrix C be of type $A_n^{(1)}$, $n \geq 2$ and $I = \{0, 1, \ldots, n\}$.
- Let us set

$$s_i(\alpha_i) = -\alpha_i, \qquad s_i(\alpha_j) = \alpha_j + \alpha_i \quad (j = i \pm 1), \qquad s_i(\alpha_j) = \alpha_j \quad (j \neq i \pm 1),$$

$$s_i(f_i) = f_i, \qquad s_i(f_j) = f_j \pm \alpha_i \, f_i^{-1} \quad (j = i \pm 1), \qquad s_i(f_j) = f_j \quad (j \neq i \pm 1),$$

$$\pi(\alpha_j) = \alpha_{j+1}, \qquad \pi(f_j) = f_{j+1}, \qquad j \in \mathbb{Z}/_{(n+1)\,\mathbb{Z}}.$$

Theorem. [Bobrova, 2024] Transformations given above are Bäcklund transformations of the $A_{2l}^{(1)}$ and $A_{2l+1}^{(1)}$ systems. Moreover, they define a birational representation of the extended affine Weyl group of type $A_n^{(1)}$, $n \geq 2$.

- Note that the shift operators are given by

$$T_1 = \pi \, s_n \, s_{n-1} \, \ldots \, s_1, \qquad T_2 = s_1 \, \pi \, s_n \, \ldots \, s_2, \qquad \ldots, \qquad T_{n+1} = s_n \, \ldots \, s_1 \, \pi. \tag{9}$$

- They satisfy the relation $T_1 \; T_2 \; \ldots \; T_{n+1} = 1$.
- Thus, any n of them form a basis for the lattice and we can define a discrete system.

Remark. Cases $n = 2$ and $n = 3$ correspond to the P_4 and P_5 equations and discrete systems labeled by d-P(E_6) and d-P(D_5) respectively. For $n = 1$ one needs to consider the P_2 equation.

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d-Painlevé equations

Overview

Agreements

- The elements q, p belong to $\mathcal{R}$ and all constant parameters labeled by greek letters $\in \mathbb{C}$.
- Usually, t is a central element, i.e. $t \in \mathcal{Z}(\mathcal{R})$, except for the P_2 and P_4 systems.
- For discrete dynamics, we will use the standard notation. Namely, for a T-action of the translation operator T, we set $T(f) = \bar{f}$ and $T^{-1}(f) = \underline{f}$.
- Regarding difference systems, we use $T^n(f) = f_n$.

Overview

- Thanks to the paper [Bershtein et al., 2023], we know that matrix Hamiltonian Painlevé systems of all types have Bäcklund transformations forming an affine Weyl group structure.
- We have reconstructed all the generators for the extended affine Weyl groups corresponding to the non-abelian Hamiltonian systems obtained in [Bobrova and Sokolov, 2023a].
- By using the translation operators as in [Sakai, 2001] or [Kajiwara et al., 2017], we have obtained the list of non-abelian discrete systems.
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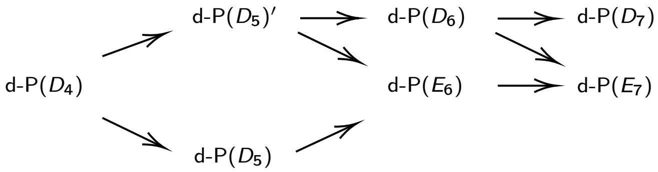

P_2 case: the system and symmetries

- Consider the P_2 system [Retakh and Rubtsov, 2010] (see also [Adler and Sokolov, 2021])

$$\begin{cases} \dot{q} &= -q^2 + p - \frac{1}{2}t, \\ \dot{p} &= qp + pq + \alpha_1. \end{cases} \qquad P_2$$

- Here we assume that t is also an element of $\mathcal{R}$ such that $\dot{t} = 1$.
- Let $\alpha_0 + \alpha_1 = 1$ and $f := -p + 2q^2 + t$.
- Its Bäcklund transformations are given below (cf. with [Bershtein et al., 2023])

	α_0	α_1	q	p	t
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q - \alpha_0 f^{-1}$	$p - 2\alpha_0 q f^{-1} - 2\alpha_0 f^{-1} q + 2\alpha_0 f^{-2}$	t
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \alpha_1 p^{-1}$	p	t
π	α_1	α_0	$-q$	$-p + 2q^2 + t$	t

- These elements form an extended affine Weyl group of type $A_1^{(1)}$:

$$\widetilde{W}(A_1^{(1)}) = \langle s_0, s_1; \pi \rangle, \\ s_i^2 = 1, \qquad \pi^2 = 1, \qquad \pi s_i = s_{i+1}\pi, \qquad i \in \mathbb{Z}/_{2\mathbb{Z}}. \tag{10}$$

P_2 case: discrete dynamics

- Consider the translation operator $T = s_1\pi$. It acts on the parameters according to the formula below and form a lattice on a line:

$$T(\alpha_0, \alpha_1) = (\alpha_0 - 1, \alpha_1 + 1). \tag{11}$$

- The q and p variables change as follows

$$\bar{q} = s_1\pi(q) = -s_1(q) = -q - \alpha_1 p^{-1}, \quad \bar{p} = s_1\pi(p) = s_1(-p + 2q^2 + t) = -p + 2\bar{q}^2 + t.$$

- So, we obtain the system

$$\begin{aligned} \bar{\alpha}_0 &= \alpha_0 - 1, & \bar{\alpha}_1 &= \alpha_1 + 1, \\ \bar{q} + q &= -\alpha_1 p^{-1}, & \bar{p} + p &= t + 2\bar{q}^2. \end{aligned} \qquad \text{d-P}(E_7)$$

- It generalizes to the non-commutative case the d-$\mathrm{P}(E_7)$ equation from [Sakai, 2001] (p. 206).
- The d-$\mathrm{P}(E_7)$ system can be rewritten in the difference form

$$\begin{cases} q_{n+1} + q_n &= -\alpha_{1,n} p_n^{-1} \\ p_n + p_{n-1} &= 2q_n^2 + t, \end{cases} \qquad \alpha_{1,n} = \alpha_1 + n, \tag{12}$$

which reduces to the following second-order difference equation:

$$\alpha_{1,n}\left(q_{n+1} + q_n\right)^{-1} + \alpha_{1,n-1}\left(q_n + q_{n-1}\right)^{-1} = -2q_n^2 - t, \quad \alpha_{1,n} = \alpha_1 + n. \qquad \text{alt-d-P}_1$$

d-P(E_7) system (1)

Lax pair

- One may also consider the corresponding discrete linear problem

$$\begin{cases} \partial_\lambda Y_n &= \mathcal{A}_n \, Y_n, \\ Y_{n+1} &= \mathcal{B}_n \, Y_n. \end{cases} \tag{13}$$

- A Lax pair for the d-P(E_7) is given by

$$\mathcal{A}_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & 1 \\ 2p & 0 \end{pmatrix} \lambda + \begin{pmatrix} -p + \frac{1}{2} t & -q \\ 2pq + 2\alpha_1 & p - \frac{1}{2} t \end{pmatrix},$$
$$\mathcal{B}_n = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -2q & -1 \\ -2\bar{p} & 0 \end{pmatrix},$$

  where $t, \ \lambda \in \mathcal{Z}(\mathcal{R})$.
- Note that the compatibility condition is satisfied, since the commutator $[p, q]$ is invariant under the map

$$\psi : \mathcal{R}^2 \to \mathcal{R}^2, \quad (q, p) \mapsto (\bar{q}, \bar{p}) = \left(-p + t + 2q^2, \, -q - \bar{\alpha}_1 (-p + t + 2q^2)^{-1}\right). \tag{14}$$

- Once $t \in \mathcal{R}$, the commutator $[p, q]$ is no longer a conserved quantity.

Remark. The latter fact might have been caused by the Hamiltonian structure similarly to the case of non-abelian Hamiltonian ODEs (see Lemma 1 in [Bobrova and Sokolov, 2023a] and its generalisation, Lemma 2.1, in [Bobrova, 2023]).

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d-P(E_7) system (2)

Hamiltonian structure

- Let us use non-abelian partial derivatives introduced in [Kontsevich, 1993].
- Similar to [Mase et al., 2020], we call a difference discrete system Hamiltonian if there exists a function $H = H(q, \bar{p})$ such that the system can be rewritten in the form

$$p = \partial_q H, \qquad \bar{q} = \partial_{\bar{p}} H. \tag{15}$$

- For the d-P(E_7) system, a Hamiltonian is $H = -q\,\bar{p} + tq + \frac{1}{3}q^3 - \bar{\alpha}_1 \ln \bar{p}$, where for the symbol $\ln f$ we define the right logarithmic derivative by $d_t(\ln f) := f^{-1}\,\dot{f}$.
- Then, the non-abelian derivatives are

$$\partial_q H = -\bar{p} + q^2 + t, \qquad \partial_{\bar{p}} H = -q - \bar{\alpha}_1\,\bar{p}^{-1}, \qquad \partial_t H = q \tag{16}$$

and (15) is equivalent to the d-P(E_7) system.

Continuous limit

- One may consider a non-abelian analog for the continuous limit. Then, by using the formulas

$$q = 1 + \varepsilon^2\,Q - \frac{1}{6}\,\varepsilon^3\,P, \quad p = -2 + 2\varepsilon^2\,Q + \frac{2}{3}\,\varepsilon^3\,P, \quad t = -6 + \frac{1}{3}\,\varepsilon^4\,T, \quad \alpha_1 = 4 + \frac{2}{3}\,\varepsilon^4\,T,$$

the d-P(E_7) has the P_1 system in the limit $\varepsilon \to 0$:

$$\dot{q} = p, \qquad \dot{p} = 6q^2 + t. \tag{P_1}$$

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Further questions

A study of the obtained d-Painlevé equations

- We expect that these equations admit Lax pairs and have a Hamiltonian form.
- We also expect that they have a continuous limit to known non-abelian differential Painlevé equations obtained in [Bobrova and Sokolov, 2023b].
- Commutative d- and differential Painlevé equations are connected with the orthogonal polynomials [Van Assche, 2022]. Orthogonal polynomials have a non-commutative analog (see [Gelfand et al., 1995]). We assume that our equations are connected with them.

Other discrete Painlevé equations

- Our method can be applied to the q-discrete Painlevé equations.
- In particular, one may define a non-abelian version for the q-P_6 equation which generalizes the matrix equation obtained in [Kawakami, 2020] to the purely non-abelian case. (an ongoing project)
- How to derive a non-commutative *ell*-discrete Painlevé equation?

Non-abelian geometry related to Painlevé equations

- What is the Okamoto space of initial data of non-abelian differential Painlevé equations?
- We would like to generalize the method of the Painlevé equations' classification introduced in the Sakai's paper [Sakai, 2001]. Recent developments might be found in [Rains, 2019].

Cluster algebras and discrete Painlevé equations

- It is known that discrete Painlevé equations are connected with claster algebras (see, e.g. [Bershtein et al., 2018]). Might we have the same connection in the non-abelian case?

Many thanks!

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