Separation of Variables of the Hitchin systems as classical limit of geometric Langlands

May 13, 2024

Integrable systems and automorphic forms

Université de Lille, France

Duong Đinh

(Max Planck Institute for Mathematics, Bonn, Germany)

based on (1) arXiv:2312.13393, with J. Teschner

(2) works to appear, with T. Figiel and J. Teschner

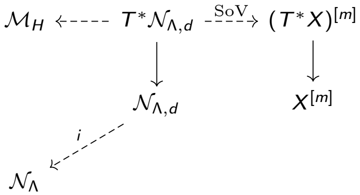
(3) D. Đ's PhD thesis

Introduction

In the first part of this talk, I will review the Hitchin moduli spaces of Higgs bundles, which are integrable systems. I will in particular give some examples in genus zero to demonstrate the role of CFT in the quantization of these integrable systems, which in turn is the key to their approach to geometric Langlands correspondence.

In the second part of the talk, I will summarize the results of my recent article arXiv:2312.13393 joint with J. Teschner. It concerns the construction of a symplectomorphism SoV, which can be regarded as the classical limit of the geometric Langlands correspondence for higher genus.

(A snapshot of where the symplectomorphism SoV is situated.)



Hitchin moduli space $\mathcal{M}_H$

Let X be a compact Riemann surface of genus ≥ 2.

- A rank-2 holomorphic bundle E is called stable if any sub-line bundle $L \hookrightarrow E$ satisfies
$$\deg(\det(E)) - 2\deg(L) > 0.$$
- An $SL_2(\mathbb{C})$-Higgs bundle on X is a pair (E, ϕ), where E is a rank-2 holomorphic bundle on X and $\phi : E \to E \otimes K$ is traceless, called the Higgs field.
- (E, ϕ) is called stable if either E is stable, or if E is destabilized by $L_E \hookrightarrow E$ then L_E is not ϕ-invariant.

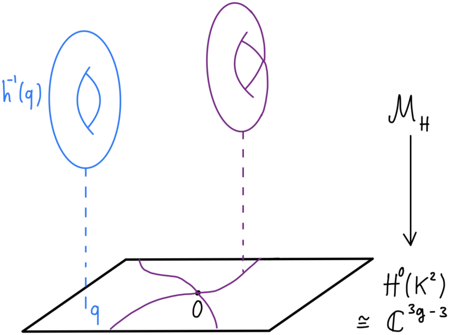
Stability condition made concrete

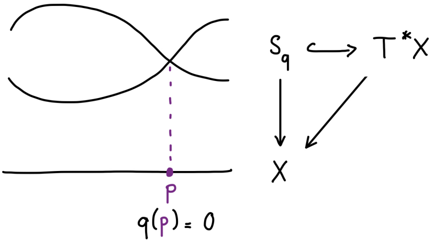
The second condition means, in local frames adapted to $L_E \hookrightarrow E$, $\phi = \begin{pmatrix} * & * \\ \phi_+ & * \end{pmatrix}$ with nonzero $\phi_+ \in H^0(KL_E^{-2}\Lambda)$, $\det(E) = \Lambda$.

The Hitchin moduli space $\mathcal{M}_H \equiv \mathcal{M}_H(X, \Lambda)$, constructed by Hitchin in 1987, is the moduli space of stable $SL_2(\mathbb{C})$-Higgs bundles on X where all underlying bundles have determinant Λ. The cotangent of the moduli space $\mathcal{N}_\Lambda$ of stable bundles is an open dense subset $T^*\mathcal{N}_\Lambda \subset \mathcal{M}_H$.

Another natural fibration, called the Hitchin fibration, is defined by the spectral data of Higgs fields

$$h : \mathcal{M}_H \to H^0(K^2),$$
$$[E, \phi] \mapsto \det(\phi).$$





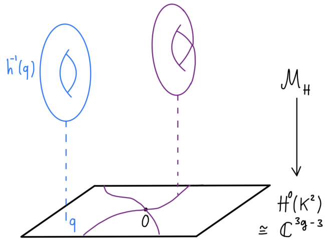
Associated to $q = \det(\phi) \in H^0(K^2)$ is a spectral curve $S_q \xrightarrow{\pi_q} X$, locally given by

$$v^2 + q(u) = 0,$$

for (u, v) being local coordinates of a trivialization $T^*X\mid_U$. The involution $S_q \xrightarrow{\sigma} S_q$ exchanges the two roots.

Hitchin moduli space $\mathcal{M}_H$

Spectral correspondence and integrable structure



- If q and S_q are non-dengerate, i.e. $\mathrm{div}(q)$ has no repeated zero, $[E, \phi] \in \mathcal{M}_H$ with $\det(\phi) = q$ is equivalent to its *eigen-line bundle*

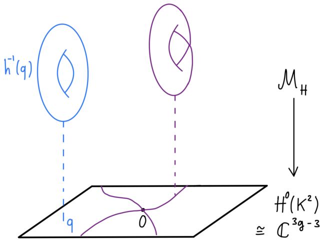
$$\mathcal{L}_{(E,\phi)} \hookrightarrow \pi^*(E).$$

- By tensoring with a fixed line bundle, Hitchin showed

$$h^{-1}(q) \cong \mathrm{Prym}(S_q)$$

where $\mathrm{Prym}(S_q) = \{[\mathcal{L}] \in \mathrm{Jac}_0(S_q) \mid \mathcal{L} \otimes \sigma^* \mathcal{L} \cong \mathcal{O}_{S_q}\}$.

Quantum Hitchin systems



By taking a basis of the Hitchin base $H^0(K^2) \simeq \mathbb{C}^{3g-3}$, one can define Poisson-commuting functions H_i on $\mathcal{M}_H$ called the classical Hitchin Hamiltonians.

Quantum Hitchin Hamiltonians

Recall that the cotangent of moduli of bundles define an open dense subset of $\mathcal{M}_H$. Beilinson-Drinfeld in 1991 used techniques inspired by CFT to show that there exist differential operators $\hat{H}_i$ on the moduli of bundles such that

- the symbols of $\hat{H}_i$ coincide with the classical Hitchin Hamiltonians H_i;
- $\left[\hat{H}_i, \hat{H}_j\right] = 0$;

These differential operators are called quantum Hitchin Hamiltonians.

Quantum Hitchin systems on punctured spheres

To demonstrate the role of CFT in the quantization of Hitchin systems, let us switch to the case $g = 0$ where computations can be explicit.

We will need to add marked points/punctures to have enough moduli, but the advantage is explicit coordinates on (most of) the moduli spaces.

Higgs bundles on punctured spheres

Let $X = \mathbb{P}^1 \setminus \{z_1, ..., z_N\}$. A "parabolic bundle" E on X then is characterized by transition functions on a punctured disk at each z_r,

$$(E)_r = \begin{pmatrix} 1 & x_r \\ 0 & 1 \end{pmatrix}$$

A Higgs field then takes the form $\phi = \sum_{r=1}^{n} \phi_r/(y - z_r)$ for

$$(E)_r \begin{pmatrix} l_r & 0 \\ p_r & -l_r \end{pmatrix} (E)_r^{-1} = \begin{pmatrix} x_r p_r + l_r & x_r^2 p_r + 2 l_r x_r \\ p_r & -x_r p_r - l_r \end{pmatrix} \equiv \begin{pmatrix} J_r^0 & J_r^- \\ J_r^+ & -J_r^0 \end{pmatrix}. \quad (0.1)$$

Three constraints induced by regularity at $y = \infty$ generate a symplectic reduction of the phase space. The classical Hitchin Hamiltonians are the expansion parameters in

$$\det(\phi) = \sum_{r=1}^{n} \left(\frac{l_r^2}{(y - y_r)^2} + \frac{H_r}{y - z_r} \right) \quad (0.2)$$

.

Example

For $X = \mathbb{P}^1 \setminus \{0, 1, \infty, z\}$ and $(x_1, ..., x_4) = (0, 1, \infty, x)$, we have $H = x(x-1)(x-z)p^2$.

Quantum Hitchin Hamiltonians on punctured spheres

The quantum Hitchin Hamiltonians in this case

$$\hat{H}_r = \sum_{s \neq r} \frac{\hat{J}_{rs}}{z_r - z_s}, \qquad \hat{J}_{rs} = \hat{J}_r^0 \hat{J}_s^0 + \frac{1}{2}(\hat{J}_r^- \hat{J}_s^+ + \hat{J}_r^+ \hat{J}_s^-)$$

can be obtained by a canonical quantization $p_r \to \partial_{x_r}$

$$\hat{J}_r^+ = \partial_{x_r}, \qquad \hat{J}_r^0 = x_r \partial_{x_r} + l_r, \qquad \hat{J}_r^+ = -x_r^2 \partial_{x_r} - 2 l_r x_r$$

The tuples $\boldsymbol{x}$ and $\mathbf{p}$ are Darboux coordinates on the classical Hitchin moduli space $\mathcal{M}_H$, with $\boldsymbol{x}$ coordinates on moduli of bundles.

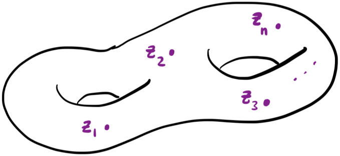
The spaces of conformal blocks

Let us now sketch how the spectral problem of the quantum Hitchin Hamiltonians

$$\hat{H}_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) = E_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}).$$

can be obtained as the "critical limit" of CFT wit Kac-Moody symmetries.

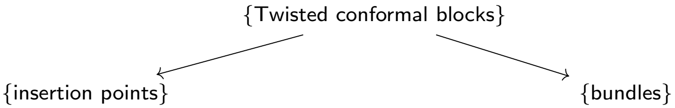
Roughly speaking, conformal blocks are building blocks of correlation functions of quantum field theories with conformal symmetry. Mathematically, they are co-invariants w.r.t. the action of the conformal symmetry on representations inserted at certain points on X.



If there are additional symmetries, such as Kac-Moody ($\hat{\mathfrak{sl}}_2$) symmetries, then it is possible that the spaces of conformal blocks have finite dimension and satisfy differential equations w.r.t. variations of the insertion points.

The spaces of conformal blocks

One can also define Kac-Moody conformal blocks "twisted" by bundles. This can be done by the so-called Kac-Moody uniformization theorem and the residue theorem. Concretely, we define the action of the "transition functions" defining a bundle by embedding its Laurent expansions at fixed points on X to $\hat{\mathfrak{sl}}_2$ (or more precisely, to the diagonal central extension of $\mathfrak{sl}_2$).



Hence we have a space of twisted conformal blocks that fiber over the moduli of bundles and moduli of curves/insertion points.

Conformal blocks and critical limits

For $\mathfrak{sl}_2$ conformal blocks on punctured spheres, they satisfy the Knizhnik–Zamolodchikov equations

$$(k-2)\partial_{z_r} F(\boldsymbol{x}, \boldsymbol{z}) = \widehat{H}_r F(\boldsymbol{x}, \boldsymbol{z})$$

that intertwine the variations of insertion points and bundles.

Critical limit of KZ equations

Using the ansatz

$$F(\boldsymbol{x}, \boldsymbol{z}) = e^{S(\boldsymbol{z})/(k-2)} \Psi(\boldsymbol{x})(1 + O(k-2)),$$

then at the critical level $k \to h^\vee = 2$, we retrieve $\Psi(\boldsymbol{x})$ as an eigen-solution of $\widehat{H}_r$ with eigenvalue $\partial_{z_r} S(\boldsymbol{z})$.

Hence $S(\boldsymbol{z})$ serves as a generating functions for the spectrum of $\widehat{H}_r$.

For punctured spheres, the RHS of KZ equations precisely are quantum Hitchin Hamiltonians. Hence the leading order of conformal blocks provides the eigenfunctions of these Hamiltonians!

Geometric Langlands

The quantization of the Hitchin system by Beilinson-Drinfeld was the key ingredient to their proof of an important case of the geometric Langlands correspondence

$$\left\{ \hat{H}_i \Psi = E_i \Psi \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{opers, i.e. collection of compatible} \\ \text{local Schrödinger equations on } X \end{array} \right\}.$$

For punctured sphere, this correspondence can be written explicitly

$$\left\{ \hat{H}_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) = E_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) \right\} \longleftrightarrow \left\{ \left[\frac{\partial^2}{\partial_z^2} - \sum_{r=1}^{n-1} \left(\frac{l_r(l_r-1)}{(z-z_r)^2} + \frac{E_r}{z-z_r} \right) \right] f(z) = 0. \right\}.$$

where the eigen-values of the quantum Hitchin Hamiltonians determine the corresponding Schrödinger equation.

Geometric Langlands made explicit

Thanks to a trick by E. Sklyanin and interpreted by E. Frenkel, this geometric Langlands correspondence can be understood as a combination of integral transform and change of variables.

1. Choose another "polarization" for quantization. Concretely, this means now we work with the Fourier transform $\Phi_{\boldsymbol{E}}(\mathbf{p})$ of the eigen-functions $\Psi_{\boldsymbol{E}}(\boldsymbol{x})$,

$$\Psi_{\boldsymbol{E}}(\boldsymbol{x}) = \int d\boldsymbol{p}\Phi_{\boldsymbol{E}}(\boldsymbol{p}) \prod_{r=1}^{n-1} e^{p_r x_r}$$

2. Let us now again consider the change of variables $\boldsymbol{p} \to \boldsymbol{u}$ defined by

$$J^{+} = \sum_{r=1}^{n-1} \frac{p_r}{y - z_r} = u_0 \frac{\prod_{k=1}^{n-3}(y - u_k)}{\prod_{r=1}^{n-1}(y - z_r)} \qquad \rightsquigarrow p_r(\boldsymbol{u}) = u_0 \frac{\prod_{k=1}^{n-3}(z_r - u_k)}{\prod_{s\neq r}^{n-1}(z_s - z_r)}$$

   We can write $\Phi_{\boldsymbol{E}}(\boldsymbol{u}) \equiv \Phi_{\boldsymbol{E}}(\boldsymbol{p}(\boldsymbol{u}))$.

Then one can show that

$$\left[\frac{\partial^2}{\partial_{u_k}^2} - \sum_{r=1}^{n-1}\left(\frac{l_r(l_r-1)}{(u_k - z_r)^2} + \frac{E_r}{u_k - z_r}\right)\right]\Phi_{\boldsymbol{E}}(\boldsymbol{u}) = 0.$$

This is the geometric Langlands correspondence made explicit, via integral transform and clever change of variables!

$$\left\{ \hat{H}_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) = E_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) \right\} \longleftrightarrow \left\{ \left[\frac{\partial^2}{\partial^2_{u_k}} - \sum_{r=1}^{n-1} \left(\frac{l_r(l_r-1)}{(u_k-z_r)^2} + \frac{E_r}{u_k-z_r} \right) \right] \Phi_{\boldsymbol{E}}(\boldsymbol{u}) = 0 \right\}$$

The key is the change of variables into zeroes of $J^+ = \sum \frac{p_r}{y-z_r}$, the lower left component of the Higgs field $\begin{pmatrix} J^0 & J^- \\ J^+ & -J^0 \end{pmatrix}$.

It is later found out [Frenkel, Gukov, Teschner '15] that there is an integral transform at the conformal field theories level which in appropriate limits reproduce the two sides of geometric Langlands on punctured spheres.

$$\{ \text{KZ equations for } H_3^+ \text{ model} \} \longleftrightarrow \{ \text{BPZ equations for Liouville field theories} \}$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\left\{ \hat{H}_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) = E_i \Psi_{\boldsymbol{E}}(\boldsymbol{x}) \right\} \longleftrightarrow \left\{ \left[\frac{\partial^2}{\partial^2_{u_k}} - \sum_{r=1}^{n-1} \left(\frac{l_r(l_r-1)}{(u_k-z_r)^2} + \frac{E_r}{u_k-z_r} \right) \right] \Phi_{\boldsymbol{E}}(\boldsymbol{u}) = 0 \right\}$$

From *Surface operators and separation of variables*, E. Frenkel, S. Gukov, J. Teschner, (2015).

E Explicit relation between Kac-Moody and Virasoro conformal blocks

We will explain in this appendix how to obtain an explicit integral transformation between the conformal blocks in Liouville theory and in the WZW model using the observations made in section 4.7. This is the separation of variables (SOV) relation (1.1) which we discussed in the Introduction.

E.1 SOV transformation for conformal blocks

In order to partially fix the global $\mathfrak{sl}_2$-constraints we shall send $z_n \to \infty$ and $x_n \to \infty$, defining the reduced conformal blocks $\check{\mathcal{Z}}^{\rm WZ}(x,z)$ which depend on $x = (x_1, \dots, x_{n-1})$ and $z = (z_1, \dots, z_{n-1})$. Let $\widetilde{\mathcal{Z}}^{\rm WZ}(\mu, z)$ be the Fourier-transformation of the reduced conformal block $\check{\mathcal{Z}}^{\rm WZ}(x,z)$ of the WZW model w.r.t. the variables x. It depends on $\mu = (\mu_1, \dots, \mu_{n-1})$ subject to $\sum_{r=1}^{n-1} \mu_r = 0$. There then exists a solution $\mathcal{Z}^{\rm L}(y,z)$ to the BPZ-equations

$$\mathcal{D}^{\rm BPZ}_{u_k} \cdot \mathcal{Z}^{\rm L} = 0, \quad \forall k = 1, \dots, l, \tag{E.1}$$

with differential operators $\mathcal{D}^{\rm BPZ}_{u_k}$ given as

$$\mathcal{D}^{\rm BPZ}_{u_k} = b^2 \frac{\partial^2}{\partial u_k^2} + \sum_{r=1}^{n} \left(\frac{\Delta_r}{(u_k - z_r)^2} + \frac{1}{u_k - z_r} \frac{\partial}{\partial z_r} \right) - \sum_{\substack{k'=1 \\ k' \neq k}}^{l} \left(\frac{3b^{-2} + 2}{4(u_k - u_{k'})^2} - \frac{1}{u_k - u_{k'}} \frac{\partial}{\partial u_{k'}} \right),$$

From *Surface operators and separation of variables*, E. Frenkel, S. Gukov, J. Teschner, (2015).

The integral transform that relates solutions of KZ and BPZ equations

$$\check{\mathcal{Z}}^{\rm WZ}(x,z) = N_J \int d\nu(u) \, \left(\textstyle\sum_{r=1}^{n-1} \lambda_r x_r\right)^J \mathcal{Z}_{\rm L}(u,z) \prod_{r=1}^{n-1} \lambda_r^{-j_r} ,$$

$$= N_J \int du_1 \dots du_{n-3} \, \mathcal{K}^{\rm SOV}(x,u) \, \mathcal{Z}^{\rm L}(u,z) ,$$

where the kernel $\mathcal{K}^{\rm SOV}(x,u)$ is defined as

$$\mathcal{K}^{\rm SOV}(x,u) := \qquad \text{(E.12)}$$

$$= \left[\sum_{r=1}^{n-1} x_r \frac{\prod_{k=1}^{n-3}(z_r - u_k)}{\prod_{s\neq r}^{n-1}(z_r - z_s)} \right]^J \prod_{k<l}^{n-3} (u_k - u_l)^{1+\frac{1}{2b^2}} \prod_{r=1}^{n-1} \left[\frac{\prod_{s\neq r}^{n-1}(z_r - z_s)}{\prod_{k=1}^{n-3}(z_r - u_k)} \right]^{\alpha_r/b} .$$

Generalization to higher genus

We would like to adapt this strategy of expressing the geometric Langlands correspondence as an integral transform

$$\left\{ \hat{H}_i \Psi = E_i \Psi \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{opers, i.e. collection of compatible} \\ \text{local Schrödinger equations on } X \end{array} \right\}.$$

to higher genus.

Let us contemplate. For genus zero, the key step is the change of variables $\boldsymbol{p} \to \boldsymbol{u}$, which are zeroes of the lower-left component of Higgs field

$$J^+ = \sum_{r=1}^{n-1} \frac{p_r}{y - z_r} = u_0 \frac{\prod_{k=1}^{n-3}(y - u_k)}{\prod_{r=1}^{n-1}(y - z_r)} \qquad \rightsquigarrow p_r(\boldsymbol{u}) = u_0 \frac{\prod_{k=1}^{n-3}(z_r - u_k)}{\prod_{s\neq r}^{n-1}(z_s - z_r)}.$$

Upon this change of variables, the Fourier transform $\Phi(\boldsymbol{u}) \equiv \Phi(\boldsymbol{p}(\boldsymbol{u}))$ of the eigenfunction Ψ satisfies the very Schrödinger equation that defines the correspondence!

So the first step of generalization to higher genus is to find a global analogue of this lower-left component. We can always do this if we choose a subbundle $L \hookrightarrow E$, and express the Higgs field in local frames adapted to L.

Generalization to higher genus

From now, we will work with a Riemann surface X of genus $g \geq 2$.

Our goal now is to construct a classical change of variables that is the analogue of the change of variables $\boldsymbol{p} \rightarrow \boldsymbol{u}$ in the punctured sphere case.

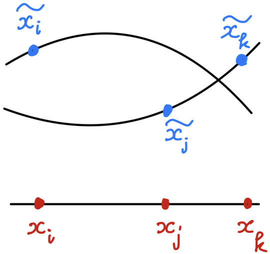
In other words, we will construct a symplectomorphism of relevant moduli spaces which can be regarded as the classical limit geometric Langlands explicit.

Baker-Akhiezer divisors

Explicit construction

Input data: (1) $(E, \phi) \in \mathcal{M}_H$ with $\det(\phi) = q$ nondegenerate, and
(2) $L \hookrightarrow E$ a subbundle.

- Suppose $\phi = \begin{pmatrix} \phi_0 & \phi_- \\ \phi_+ & -\phi_0 \end{pmatrix}$ in adapted local frames.
- Let $\sum_{r=1}^{m} x_r$ be the zero divisor of $\phi_+ \in H^0(KL^{-2}\Lambda)$.
- At each such x_r, the component(s) of $\pi^{-1}(x_r)$ are invariantly labeled by $\pm\phi_0(x_r)$. Let $\tilde{x}_r$ be the point labeled by $\phi_0(x_r)$.
- We say $D = \sum_{r=1}^{m} \tilde{x}_r$ is the Baker-Akhiezer divisor associated to $(L \hookrightarrow E, \phi)$.



Note that D does not contain the pull-back of a divisor on X.

Baker-Akhiezer divisors

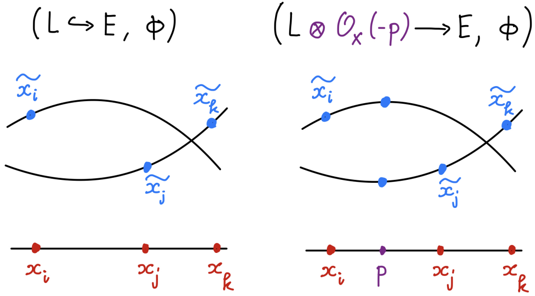
The eigen-line bundle $\mathcal{L}_{(E,\phi)}$ is a subbundle of $\pi^*(E)$ and defines s.e.s.

$$0 \longrightarrow \mathcal{L}_{(E,\phi)} \longrightarrow \pi^*(E) \longrightarrow \mathcal{L}^{-1}_{(E,\phi)}\pi^*(\Lambda) \longrightarrow 0$$

Formal definition

Given $(L \to E, \phi)$ with $\det(\phi) = q$ nondegenerate, the Baker-Akhiezer divisor associated to $(L \to E, \phi)$ is the zero divisor of the composition $\pi^*(L) \to \pi^*(E) \to \mathcal{L}^{-1}_{(E,\phi)}\Lambda$.

Such divisors have been briefly discussed already by Hitchin in his original work.



Theorem

Let $S_q \xrightarrow{\pi} X$ be the spectral curve associated to a non-degenerate quadratic differential $q \in H^0(K^2)$.

- If D is the Baker-Akhiezer divisor of $(L \to E, \phi)$, then there exists $D' < D$ with $D' = \pi^*(\mathbf{p})$ for some divisor $\mathbf{p}$ on X if and only if $L \to E$ vanishes at $\mathbf{p}$. The eigen-line bundle of (E, ϕ) is
$$\mathcal{L}_{(E,\phi)} \cong \pi^*(LK^{-1}) \otimes \mathcal{O}_{S_q}(\sigma(D)).$$
- The construction of Baker-Akhiezer divisors can be inverted and hence defines a bijection
$$\left\{ [(L \to E, \phi)] \;\middle|\; \begin{array}{l} \det(E) = \Lambda, \\ \det(\phi) = q \end{array} \right\} \longleftrightarrow \left\{ ([L], D) \;\middle|\; \begin{array}{l} D \text{ effective on } S_q, \\ KL^{-2}\Lambda \cong \mathcal{O}_X(\pi(D)) \end{array} \right\}.$$
Forgetting the line bundle induces a $2^{2g} : 1$ map $[(L \to E, \phi)] \mapsto D$.

Proof sketch

Proof sketch for part 1

1. The key ingredient in the proof is the observation that $\begin{pmatrix} v+\phi_0 \\ \phi_+ \end{pmatrix}$ defines eigen-vectors of ϕ and hence local sections of $\mathcal{L}_{(E,\phi)}$.
2. Explicit computation shows outside its zero divisor, which is precisely D, this section transforms as a section of $\pi^*(LK^{-1})$.

Terminology: Baker-Akhiezer divisor

Since $\begin{pmatrix} v+\phi_0 \\ \phi_+ \end{pmatrix}$ resembles the BA-function in the integrable system literature, we call its zero divisor "Baker-Akhiezer divisor".

Example: Hitchin section

Given a non-degenerate quadratic differential q, the Higgs bundle $[E,\phi]$ with $E = K^{1/2} \oplus K^{-1/2}$ and $\phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$ is contained in the Hitchin fiber $h^{-1}(-q)$. The Baker-Akhiezer divisors corresponding to $(K^{1/2} \hookrightarrow E, \phi)$ and $(K^{-1/2} \hookrightarrow E, \phi)$ are respectively the trivial divisor and the ramification divisor of $S_{-q} \xrightarrow{\pi} X$.
Either way, one can check $\mathcal{L}_{(E,\phi)} \cong \pi^*(K^{-1/2})$ by noting that $\pi^*(K)$ can be represented by the ramification divisor.

The moduli spaces $\mathcal{N}_{\Lambda,d}$

The moduli space $\mathcal{N}_{\Lambda,d}$ of pairs (E, L) has a simple description as a projective fibration over the moduli space Pic^d of line bundles of degree d.

$$\begin{array}{ccccc} & & \mathcal{N}_{\Lambda,d} & & \\ & \overset{i}{\swarrow} & & \overset{j}{\searrow} & \\ \mathcal{N}_\Lambda & & & & \mathrm{Pic}^d \end{array} .$$

A point (E, L) in $\mathcal{N}_{\Lambda,d}$ can be represented by an extension of the form

$$0 \to L \to E \to L^{-1}\Lambda \to 0.$$

The fiber over $L \in \mathrm{Pic}^d$ consists of classes of all such extensions, which is parameterized by $H^1(L^2\Lambda^{-1})$, plus modulo scaling

$$j^{-1}(L) = \mathbb{P}H^1(L^2\Lambda^{-1}).$$

The cotangent $T^*\mathcal{N}_{\Lambda,d}$

$$\begin{array}{ccc} & \mathcal{N}_{\Lambda,d} & \\ {}^{i}\swarrow & & \searrow^{j} \\ \mathcal{N}_\Lambda & & \mathrm{Pic}^d \end{array}.$$

The fibration structure together with Serre duality allows us to locally make a splitting at $(E,L) \in \mathcal{N}_{\Lambda,d}$ into

$$T^*_{(E,L)}\mathcal{N}_{\Lambda,d} \simeq H^0(K) \oplus \ker((L,E)) \subset H^0(K) \oplus H^0(KL^{-2}\Lambda).$$

We can equip coordinates on $\mathcal{N}_{\Lambda,d}$ by choosing reference divisors and local coordinates as follows.

- Let $\boldsymbol{q} = \sum_{i=1}^{g} q_i$ and $\check{\boldsymbol{q}} = \sum_{i=1}^{g-d} \check{q}_i$ be reduced divisors with $L \simeq \mathcal{O}_X(\boldsymbol{q} - \check{\boldsymbol{q}})$. Then local coordinates $\boldsymbol{z} = (z_1, \ldots, z_g)$ in the neighborhood of $q_1, \ldots, q_g$ provides coordinates on Pic^d.
- Projective coordinates $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{C}^N$ on the fibers of j are provided by choosing reduced divisor $\boldsymbol{p} = p_1 + \ldots + p_N$ and defining bundles by transition functions

$$\begin{pmatrix} z_i - z_i(q_i) & 0 \\ 0 & \frac{1}{z_i - z_i(q_i)} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\check{z}_j - \check{z}_j(\check{q}_j)} & 0 \\ 0 & \check{z}_j - \check{z}_j(\check{q}_j) \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{x_r}{w_r - w_r(p_r)} \\ 0 & 1 \end{pmatrix}.$$

The data $(\boldsymbol{z}, \boldsymbol{x})$ actually define local coordinates of $\mathcal{M}_{\Lambda,d}$, the moduli space of data $(L \hookrightarrow E)$. $\mathcal{M}_{\Lambda,d}$ admits a natural $\mathbb{C}^*$-action, with $\mathcal{N}_{\Lambda,d}$ its projectivization.

Proposition

The symplectic reduction of $T^*\mathcal{M}_{\Lambda,d}$ is isomorphic to an open dense subset of $T^*\mathcal{N}_{\Lambda,d}$.

An $SL_2(\mathbb{C})$-Higgs field ϕ on a bundle E with the transition functions defined w.r.t. the reference divisors and local coordinates is now equivalent to a matrix of Abelian differentials

$$\begin{pmatrix} \phi_0 & \phi_- \\ \phi_+ & -\phi_0 \end{pmatrix}.$$

Let $(\check{\boldsymbol{z}}, \boldsymbol{k})$ be the canonical conjugate coordinates of $(\boldsymbol{z}, \boldsymbol{x})$. Denote by $\check{\boldsymbol{z}}(\phi)$ and $\boldsymbol{k}(\phi)$ the evaluation of these coordinates on the pull-back of $(E, \phi) \in T^*\mathcal{N}_\Lambda$ to $T^*_{(L,E)}\mathcal{M}_{\Lambda,d}$.

Proposition

The pull-back of (E, ϕ) captures the lower-triangular part of $\begin{pmatrix} \phi_0 & \phi_- \\ \phi_+ & -\phi_0 \end{pmatrix}$ in the sense that

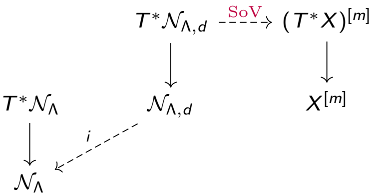
$$\begin{aligned} \check{z}_i(\phi) &= -2\phi_0(q_i), \quad && i = 1, \ldots, g, \\ k_r(\phi) &= \phi_+(p_r), && r = 1, \ldots, N. \end{aligned}$$

The map SoV

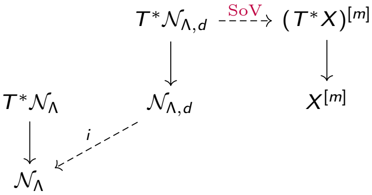
Since the explicit definitions of BA divisors make use of only this lower-left triangular part, we can define a rational map

$$SoV : T^*\mathcal{N}_{\Lambda,d} \to (T^*X)_s^{[m]}$$

which we call *Separation of Variables*. (The terminology is inspired by recalling how it resembles the classical change of variables underlying Sklyanin's trick in Gaudin model.) Here $m = \deg(KL^{-2}\Lambda)$.



The map SoV



Theorem

The restriction of *SoV* to a sufficiently small neighborhood of a generic point in $T^*\mathcal{N}_{\Lambda,d}$ is a symplectomorphism.

Proof idea

Use the prime forms on a fundamental domain of X to express the 1-forms ϕ_0 and ϕ_+ in terms of the information of their zeroes and poles. This is the high genus analogue of how rational functions on the spheres can be expressed in terms of polynomials.

Summary

- We have given examples on punctured spheres to demonstrate how Sklyanin's SoV makes the geometric Langlands correspondence explicit as an integral transform. One step of Sklyanin's strategy is a canonical transformation for the classical phase spaces, which can be regarded as the classical limit of the geometric Langlands correspondence.
- For higher genus, to generalize Sklyanin's strategy, the appropriate moduli spaces to consider are $\mathcal{N}_{\Lambda,d}$ and $\mathcal{M}_{\Lambda,d}$. The local symplectomorphism SoV we construct using Baker-Akhiezer divisors is the high genus analogue of this canonical transformation, hence can also be thought of as a classical limit of geometric Langlands.

Outlooks

- Characterize wobbly bundles in terms of Baker-Akhiezer divisors.
- Lifts of the classical and quantum Hitchin systems to $\mathcal{N}_{\Lambda,d}$.
- Analogue of Baker-Akhiezer divisors for holomorphic connections and at the CFT level.
- Higher genus generalization of geometric Langlands as integral transform.
- The role of Hecke modifications.
- Hausel-Thaddeus' mirror symmetry.
- T. Bridgeland's program: Donaldson-Thomas invariants, moduli spaces of Higgs bundles and connections, Joyce structure on space of quadratic differentials.