Integrable Lagrangians, modular forms and degenerations

E.V. Ferapontov

Department of Mathematical Sciences, Loughborough University, UK

E.V.Ferapontov@lboro.ac.uk

Based on ongoing project with S. Opanasenko

Integrable Systems and Automorphic Forms

University of Lille, 13-16 May 2024

Plan:

- Example: classification of solutions of the Chazy equation
- General problem: degenerations of modular forms
- 3D integrable Lagrangians $\int f(v_{x_1}, v_{x_2}, v_{x_3})\, dx$
  - Examples and known results
  - Generic Lagrangians via Picard modular forms
  - Classification of non-generic Lagrangians
  - Dispersionless Lax pairs

Based on:

[1] E.V. Ferapontov, K.R. Khusnutdinova and S.P. Tsarev, On a class of three-dimensional integrable Lagrangians, Comm. Math. Phys. **261**, no. 1 (2006) 225-243.

[2] E.V. Ferapontov and A. V. Odesskii, Integrable Lagrangians and modular forms, Journal of Geometry and Physics **60**, no. 6-8 (2010) 896-906.

[3] F. Cléry, E.V. Ferapontov, A. Odesskii, D. Zagier, Integrable Lagrangians and Picard modular forms, work in progress.

[4] S. Opanasenko, E.V. Ferapontov, Differential equations for modular forms and Jacobi forms; arXiv:2212.01413.

Example: classification of solutions of the Chazy equation

Chazy equation:

$$f''' + 2ff'' - 3(f')^2 = 0.$$

$SL(2,\mathbb{R})$ symmetry:

$$\tilde{z} = \frac{az+b}{cz+d}, \quad \tilde{f} = (cz+d)^2 f + 12c(c\tau + d).$$

Generic solution:

$$f(z) = E_2(iz/\pi) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) e^{-2nz},$$

where E_2 is the Eisenstein series, which is a quasi-modular form of weight 2 on $SL(2,\mathbb{Z})$ and $\sigma_1(n)$ denotes the sum of all positive divisors of n. This solution belongs to the open three-dimensional $SL(2,\mathbb{R})$-orbit.

There are also two non-generic orbits of dimensions two and one,

$$f(z) = \frac{1}{(cz-a)^2} + \frac{6c}{cz-a} \qquad \text{and} \qquad f(z) = \frac{6c}{cz-a},$$

the orbits of constant solutions $f(z) = 1$ and $f(z) = 0$, respectively.

General problem: degenerations of modular forms

Every modular form f (classical, Jacobi, Siegel, Picard, ...) on a discrete subgroup Γ of a Lie group G solves a nonlinear PDE system Σ such that:

- system Σ is involutive (compatible);
- system Σ is of finite type (has finite-dimensional solution space);
- system Σ is G-invariant, furthermore, the Lie group G acts on the solution space of Σ locally transitively with an open orbit (thus, the dimension of the solution space of Σ equals $\dim G$);
- the modular form f is a generic solution of system Σ (f belongs to the open orbit), in particular, solution f has discrete stabiliser Γ;
- system Σ is expressible via differential invariants of a suitable action of G.

In the case of classical modular forms f, we have: $\Gamma = \text{SL}(2, \mathbb{Z})$, $G = \text{SL}(2, \mathbb{R})$, system Σ is a third-order nonlinear $\text{SL}(2, \mathbb{R})$-invariant ODE for f.

General problem: classify solutions of Σ corresponding to non-generic orbits.

Non-generic solutions will automatically have continuous symmetries from G and can be obtained using the well-developed machinery of symmetry analysis of differential equations.

3D integrable Lagrangians $\int f(v_{x_1}, v_{x_2}, v_{x_3})\, dx$: examples

Euler-Lagrange equation:

$$(f_{v_{x_1}})_{x_1} + (f_{v_{x_2}})_{x_2} + (f_{v_{x_3}})_{x_3} = 0.$$

Example 1. Linear wave equation:

$$v_{x_1x_1} - v_{x_2x_2} - v_{x_3x_3} = 0, \qquad f = v_{x_1}^2 - v_{x_2}^2 - v_{x_3}^2.$$

Example 2. Dispersionless Kadomtsev-Petviashvili equation:

$$v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0, \qquad f = v_{x_1}v_{x_2} - \frac{1}{3}v_{x_1}^3 - v_{x_2}^2.$$

Example 3. Boyer-Finley equation:

$$v_{x_1x_1} + v_{x_2x_2} - e^{v_{x_3}}v_{x_3x_3} = 0, \qquad f = v_{x_1}^2 + v_{x_2}^2 - 2e^{v_{x_3}}.$$

Example 4. Equation

$$v_{x_1}v_{x_2x_3} + v_{x_2}v_{x_1x_3} + v_{x_3}v_{x_1x_2} = 0, \qquad f = v_{x_1}v_{x_2}v_{x_3}.$$

Modular example

Example 5. Lagrangian density $f = v_{x_1} v_{x_2} g(v_{x_3})$ generates the Euler-Lagrange equation

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + \left(v_{x_1} v_{x_2} g'(v_{x_3})\right)_{x_3} = 0.$$

The requirement of integrability leads to a fourth-order ODE for $g(z)$:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

The generic solution $g(z)$ of this ODE can be represented in the form

$$g(z) = \sum_{(k,l)\in\mathbb{Z}^2} e^{2\pi i(k^2+kl+l^2)z} = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + \dots.$$

where $q = e^{2\pi i z}$. It is a modular form of weight one and level three (with a character), known as the Eisenstein series $E_{1,3}(z)$. The generic solution $g(z)$ also satisfies a third-order ODE,

$$g^2 g'''^2 - 6g'(3gg'' - 4g'^2)g''' + 18gg''^3 - (g^6 + 27g'^2)g''^2 + 4g^5 g'^2 g'' - 4g^4 g'^4 = 0;$$

note that the fourth-order ODE is a differential consequence of the third-order ODE.

The function $g(z) = z$ is a non-generic solution of the above fourth-order ODE; it corresponds to the Lagrangian density $f = v_{x_1} v_{x_2} v_{x_3}$. Note that $g(z) = z$ does not satisfy the third-order ODE.

Integrable Lagrangians: summary of known results

- There exist three approaches to integrability of 3D Euler-Lagrange equations of the above type (method of hydrodynamic reductions, method of dispersionless Lax pairs, method of integrable conformal geometry), leading to the same integrability conditions and classification results.

- Integrability conditions for a Lagrangian density f form a fourth-order involutive PDE system Σ expressing all fourth-order partial derivatives of f in terms of lower-order derivatives. The system of integrability conditions is invariant under a 20-dimensional Lie group G that acts on the 20-dimensional parameter space of integrable Lagrangians with an open orbit.

- Master-Lagrangian corresponding to the open orbit is expressible via (vector-valued) Picard modular forms on a discrete subgroup Γ of G.

Integrability conditions

Integrability conditions form a PDE system Σ for the Lagrangian density $f(v_{x_1}, v_{x_2}, v_{x_3})$:

$$d^4f = d^3f\,\frac{dH}{H} + \frac{3}{H}\det(dM),$$

where d^3f and d^4f are the symmetric differentials of f; the Hessian H and the 4×4 matrix M are defined as

$$H = \det\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}, \quad M = \begin{pmatrix} 0 & f_x & f_y & f_z \\ f_x & f_{xx} & f_{xy} & f_{xz} \\ f_y & f_{xy} & f_{yy} & f_{yz} \\ f_z & f_{xz} & f_{yz} & f_{zz} \end{pmatrix}.$$

Here $(x, y, z) = (v_{x_1}, v_{x_2}, v_{x_3})$. System Σ is in involution and its solution space is 20-dimensional. The 20-dimensional symmetry group G of system Σ consists of projective transformations of x, y, z and linear transformations of f:

$$\tilde{x} = \frac{l_1(x,y,z)}{l_0(x,y,z)}, \quad \tilde{y} = \frac{l_2(x,y,z)}{l_0(x,y,z)}, \quad \tilde{z} = \frac{l_3(x,y,z)}{l_0(x,y,z)}, \quad \tilde{f} = \frac{f}{l_0(x,y,z)},$$

$$\tilde{f} = \epsilon f + \alpha x + \beta y + \gamma z + \delta.$$

Generic Lagrangian density f?

Weierstrass sigma function σ and integers C_k

Let σ be the Weierstrass sigma function (equianharmonic case $g_2 = 0$). It solves the ODE

$$\sigma\sigma'''' - 4\sigma'\sigma''' + 3\sigma''^2 = 0$$

and possesses a power series expansion

$$\sigma(z) = \sum_{k\geq 0} C_k \frac{z^{6k+1}}{(6k+1)!}$$

where C_k are certain integers:

$$1,\ 1,\ -6,\ -552,\ 18600,\ -9831240,\ \ldots.$$

These integers will feature in the formulas for the generic Lagrangian density f.

Lagrangian density $f = v_{x_1} v_{x_2} \, g(v_{x_3})$

Euler-Lagrange equation:

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + \big(v_{x_1} v_{x_2} g'(v_{x_3})\big)_{x_3} = 0.$$

Integrability condition:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

Below we give three equivalent representations of the generic g:

Theta representation;

Power series representation;

Parametric representation.

Auxiliary hypergeometric equation

Consider an auxiliary hypergeometric equation,

$$u(1-u)h_{uu} + (1-2u)h_u - \frac{2}{9}h = 0.$$

The geometry behind this equation is a one-parameter family of genus 2 trigonal curves,

$$r^3 = t(t-1)(t-u)^2,$$

supplied with the holomorphic differential $\omega = dt/r$. The corresponding periods, $h = \int_a^b \omega$ where $a, b \in \{0, 1, \infty, u\}$, form a 2-dimensional vector space and satisfy the above (Picard-Fuchs) hypergeometric equation.

Generic $g(z)$

Generic solution of the ODE

$$g''''(g^2g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3g''' - g^2(g''')^2 = 0$$

has three equivalent representations.

1. **Theta representation:**

$$g(z) = \sum_{(k,l)\in\mathbb{Z}^2} e^{2\pi i(k^2+kl+l^2)z} = 1 + 6q + 6q^3 + 6q^4 + 12q^4 + ...,$$

   which is the Eisenstein series $E_{1,3}(z)$.

2. **Power series:**

$$g(z) = \sum_{k\geq 0} C_k^2 \frac{z^{6k+1}}{(6k+1)!}$$

   where the integers C_k appear in the power series expansion of the Weierstrass σ-function.

3. **Parametric form:**

$$z = \frac{h_1(u)}{h_2(u)}, \quad g = h_2(u)$$

   where h_1, h_2 are two linearly independent solutions of the auxiliary hypergeometric equation.

Generic Lagrangian density $f(x, y, z)$

Generic Lagrangian density possesses, among others, a remarkable power series representation,

$$f(x, y, z) = \sum_{i,j,k \geq 0} C_i C_j C_k C_{i+j+k} \frac{x^{6i+1}}{(6i+1)!} \frac{y^{6j+1}}{(6j+1)!} \frac{z^{6k+1}}{(6k+1)!},$$

where the integers C_k appear in the power series expansion of the Weierstrass σ-function.

Let us now turn to non-generic Lagrangians.

Classification of non-generic Lagrangian densities f

Orbit representative (12 cases)	Orbit dimension
$f = f(v_{x_1}, v_{x_2}, v_{x_3})$	20
$f = u_{x_1} g(v_{x_2}, v_{x_3})$	19
$f = v_{x_1} v_{x_3} - 2\sigma(v_{x_1}) \mathrm{e}^{v_{x_2}}$	18
$f = v_{x_1} v_{x_2} g(v_{x_3})$	18
$f = v_{x_1} v_{x_2} v_{x_3}$	17
$f = v_{x_1} v_{x_3} - 2v_{x_1} \mathrm{e}^{v_{x_2}}$	17
$f = v_{x_1} v_{x_3} - 2v_{x_1} v_{x_2}^2 - 2v_{x_1}^7$	17
$f = v_{x_1} v_{x_3} - 2v_{x_1} v_{x_2}^2$	16
$f = v_{x_1} v_{x_3} - 2\mathrm{e}^{v_{x_2}}$	16
$f = v_{x_1} v_{x_3} - 2v_{x_2}^2 - 2v_{x_1}^2 v_{x_2} + \frac{1}{2} v_{x_1}^4$	16
$f = v_{x_1} v_{x_3} - v_{x_2}^2 - \frac{1}{3} v_{x_3}^3$	15
$f = v_{x_1}^2 - v_{x_2}^2 - v_{x_3}^2$	13

(Possibly incomplete)

Dispersionless Lax pairs

We say that a pair of Hamilton-Jacobi type equations for an auxiliary function S,

$$S_{x_3} = f(S_{x_1}, v_{x_1}, v_{x_2}, v_{x_3}), \quad S_{x_2} = g(S_{x_1}, v_{x_1}, v_{x_2}, v_{x_3}),$$

constitutes a dispersionless Lax representation of a given equation for v iff the compatibility condition $S_{x_3x_2} = S_{x_2x_3}$ is equivalent to this equation. Lax pairs of this form have been introduced by Zakharov as dispersionless limits of Lax pairs of integrable soliton equations.

Example. The dispersionless Kadomtsev-Petviashvili equation,

$$v_{x_1x_3} - v_{x_1}v_{x_1x_1} - v_{x_2x_2} = 0, \qquad f = v_{x_1}v_{x_2} - \frac{1}{3}v_{x_1}^3 - v_{x_2}^2,$$

possesses the Lax pair

$$S_{x_3} = \frac{1}{3}S_{x_1}^3 + v_{x_1}S_x + v_{x_2}, \qquad S_{x_2} = \frac{1}{2}S_{x_1}^2 + v_{x_1}.$$

Parametric Lax pairs

In some cases it is more convenient to work with parametric Lax pairs,

$$S_{x_1} = f(p, v_{x_1}, v_{x_2}, v_{x_3}), \quad S_{x_2} = g(p, v_{x_1}, v_{x_2}, v_{x_3}), \quad S_{x_3} = h(p, v_{x_1}, v_{x_2}, v_{x_3}),$$

where p is a parameter. In the latter case, the compatibility condition takes the form

$$f_p(g_{x_3} - h_{x_2}) + g_p(h_{x_1} - f_{x_3}) + h_p(f_{x_2} - g_{x_1}) = 0.$$

Parametric Lax pairs have appeared in the context of the universal Whitham hierarchy.

Example. The equation

$$v_{x_1} v_{x_2 x_3} + v_{x_2} v_{x_1 x_3} + v_{x_3} v_{x_1 x_2} = 0, \qquad f = v_{x_1} v_{x_2} v_{x_3},$$

possesses the parametric Lax pair

$$\frac{S_{x_1}}{v_{x_1}} = \zeta(p), \quad \frac{S_{x_2}}{v_{x_2}} = \zeta(p) + \frac{\wp'(p) + \lambda}{2\wp(p)}, \quad \frac{S_{x_3}}{v_{x_3}} = \zeta(p) + \frac{\wp'(p) - \lambda}{2\wp(p)},$$

where ζ and $\wp$ are the Weierstrass functions, $\zeta' = -\wp, \;\; \wp'^2 = 4\wp^3 + \lambda^2$.

Lax pair for the Lagrangian density $f = v_{x_1} v_{x_2}\, g(v_{x_3})$

Euler-Lagrange equation:

$$(v_{x_2} g(v_{x_3}))_{x_1} + (v_{x_1} g(v_{x_3}))_{x_2} + \left(v_{x_1} v_{x_2} g'(v_{x_3})\right)_{x_3} = 0.$$

Integrability condition:

$$g''''(g^2 g'' - 2g(g')^2) - 9(g')^2(g'')^2 + 2gg'g''g''' + 8(g')^3 g''' - g^2(g''')^2 = 0.$$

Parametric Lax pair:

$$S_{x_1} = v_{x_1}\varphi(p, v_{x_3}), \quad S_{x_2} = v_{x_2}\psi(p, v_{x_3}), \quad S_t = \eta(p, v_{x_3}).$$

System for φ, ψ, η:

$$\begin{aligned} \psi_p\eta_z - \psi_z\eta_p - \varphi\psi_p &= \frac{g'}{g}(\varphi - \psi)\eta_p, \\ \eta_p\varphi_z - \eta_z\varphi_p + \psi\varphi_p &= \frac{g'}{g}(\varphi - \psi)\eta_p, \\ \varphi_p\psi_z - \varphi_z\psi_p &= \frac{g''}{2g}(\varphi - \psi)\eta_p. \end{aligned}$$

Particular solution:

$$\varphi = \alpha_1(z)\eta, \qquad \psi = \alpha_2(z)\eta, \qquad \alpha_{1,2} = \frac{1}{6}\left(-\frac{(gg'' - 2g'^2)'}{gg'' - 2g'^2} \pm g^2\right).$$

Linear (non-generic) Lax pair:

$$S_{x_1} = \alpha_1(v_{x_3})\, v_{x_1} S_{x_3}, \qquad S_{x_2} = \alpha_2(v_{x_3})\, v_{x_2} S_{x_3}.$$