Irrationality of the moduli spaces of polarised generalised Kummer varieties

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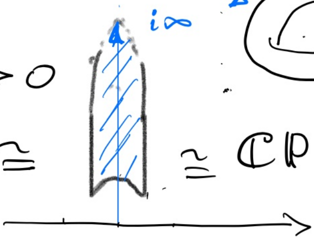
1. Itroduction: moduli of elliptic curves

$$E_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$$

$$\tau = x+iy \in \mathbb{H}^+, \quad y>0$$

$$SL_2(\mathbb{Z})\backslash\mathbb{H}^+ \cong$$



$$\cong \mathbb{CP}^1$$

$f(\tau)\,d\tau$ – голом. диф. форма на $SL_2(\mathbb{Z})\backslash\mathbb{H}$, если

1) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-2} f(\tau)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

2) $f(\tau) = \sum_{n\geq 0} a(n)\, q^n$, $q = e^{2\pi i \tau}$, $\boxed{a_0 = 0}$

$$d\tau = \frac{dq}{q}$$

$f(i\infty) = 0$

2. Modular varieties of orthogonal type

Let's replace the imaginary axis $y > 0$ with the cone of future of an integral hyperbolic lattice of signature $(1, n-1)$

$$V^+(L_{1,n-1}) = \{Y \in L_{1,n-1} \otimes \mathbb{R},\ (Y, Y) > 0\}^+.$$

For the lattice $L = U \oplus L_{1,n-1}$ of signature $(2, n)$, where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the hyperbolic plane we define two realisations of the hermitian homogeneous domain of type IV

$$\Omega(L) = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) : (Z, Z) = 0, (Z, \bar{Z}) > 0\}^+ \cong$$

$$\mathcal{D}(L) = \{X + iY, X \in L_{1,n-1} \otimes \mathbb{R}, Y \in V^+(L_{1,n-1})\}.$$

The integral orthogonal group $O^+(L)$ of the lattice L acts on $\Omega(L)$.

For any lattice L of signature $(2, n)$ and any subgroup of finite index $\Gamma < O^+(L)$ we define the modular variety of dimension n

$$\mathcal{M}_\Gamma(L) = \Gamma \setminus \Omega(L), \qquad \dim_{\mathbb{C}} \mathcal{M}_\Gamma(L) = n.$$

3. Modular varieties of orthogonal type

$\operatorname{sign}(L) = (2,3)$ – the moduli spaces of polarised Abelian surfaces (Gr-1993) and polarised Kummer surfaces.

$\operatorname{sign}(L) = (2,19)$ – the moduli spaces of polarised K3 surfaces. The last open question of A.Weil's program (1956) on K3 surfaces (GHS-2007).

$\operatorname{sign}(L) = (2,20)$ – the moduli spaces of polarised hyperkähler varieties of type $Hilb^n(K3)$ (GHS-2010, 2014).

$\operatorname{sign}(L) = (2,21)$ – the moduli spaces of polarised hyperkähler varieties of OG10 (GHS-2011).

$\operatorname{sign}(L) = (2,10)$ – the moduli spaces of Enriques surfaces and of polarised Enriques surfaces.

$\operatorname{sign}(L) = (\mathbf{2},\mathbf{4})$ **– the moduli spaces of polarised hyperkähler varieties of type** $Kum^n(A)$. There are no concrete results on the geometric type of these moduli spaces.

4. IHSVs

A **K3 surface** is the first example of irreducible holomorphic symplectic varieties (or hyperkähler manifold).

Def. A compact Kähler manifold X is called an irreducible holomorphic symplectic manifold (or **hyperkähler manifold**) if

1) X is simply-connected,
2) $H^0(X, \Omega^2_X) = H^{2,0}(X) = \mathbb{C}\omega$ where ω is an everywhere nondegenerate holomorphic 2-form.

Beauville (1983) found two infinite series of IHSVs:

1) The length n Hilbert scheme $Hilb^n(S)$ for a $K3$ surface S, and its deformations.

2) Let A be a 2-dimensional complex torus and consider $A^{[n+1]} = Hilb^{n+1}(A)$ $(n \geq 2)$ with the morphism of $p : A^{[n+1]} \to A$ given by addition. Then $X = p^{-1}(0)$ is an IHSV, called a **generalised Kummer variety**. The deformation space of these manifolds has dimension 5.

5. Polarisations

A great deal of information on IHSVs is encoded in the cohomology group $H^2(X,\mathbb{Z})$ coming with the structure of an integral quadratic lattice, the Beauville-Bogomolov form.

$X \sim K3^{[n]}$: $\quad H^2(X,\mathbb{Z}) \cong 3U \oplus 2E_8(-1) \oplus \langle -2(n-1)\rangle$.

$X \sim Kum^n$: $\; H^2(X,\mathbb{Z}) \cong 3U \oplus \langle -2(n+1)\rangle \;\; (n \geq 2)$.

This isomorphism of the lattices is called **marking** of X.

A **polarisation** of X is a choice of ample line bundle $\mathcal{L}$ on X

$$c_1(\mathcal{L}) = h \in H^2(X,\mathbb{Z}) \cong L_{BB}, \quad h^2 = 2d > 0.$$

We have $(h, L) = div(h)\,\mathbb{Z}$. If div(h)=1, then the polarisation is called **split**. For a split polarisation we get a lattice of signature $(2,4)$ (for GKVs)

$$h_L^{\perp} = L_{2n+2,2d} \cong 2U \oplus \langle -2(n+1)\rangle \oplus \langle -2d\rangle.$$

A period of X is defined as follows

$$[\varphi(\omega)] \in \{[Z] \in \mathbb{P}(L_{2n+2,2d} \otimes \mathbb{C}) : (Z,Z) = 0, (Z,\bar{Z}) > 0\}.$$

6. The moduli space $\mathcal{M}(Kum^n, h_{2d}^{split})$

Global Torelli Theorem is true for IHSVs (Verbitski, Markmann). In our case we have an open immersion

$$\mathcal{M}(Kum^n, h_{2d}^{split}) \to \Gamma_{2n+2,2d} \setminus \Omega(L_{2n+2,2d}).$$

The group $\Gamma_{2n+2,2d}$ is the group of Markmann's parallel transport operators acting on $H^2(X, \mathbb{C})$. Montgardi (2018) proved that

$$\Gamma_{2n+2,2d} = \langle \widetilde{SO}^+(L_{2n+2,2d}), \sigma_{-(2n+2)} \rangle,$$

where $\sigma_{-(2n+2)}$ is the reflection with respect to the generator of $\langle -2(n+1) \rangle$ of the lattice $L_{2n+2,2d}$ of signature $(2,4)$. "Tilde" means the stable orthogonal group acting trivially on the finite discriminate group $L^*_{2n+2,2d}/L_{2n+2,2d}$ of order $(2n+2) \cdot 2d$.

7. Canonical differential forms on $\mathcal{M}(Kum^n, h_{2d}^{split})$

Let $F_4(\tau, z_1, z_2, \omega)$ be a modular form of weight 4 on $\mathcal{D}(L_{2n+2,2d})$ with respect to $\Gamma_{2n+2,2d}$ with character $det : \Gamma_{2n+2,2d} \to \{\pm 1\}$.

Then $F_4(Z)dZ$ is a canonical differential form on an **open part** of the modular variety. It can be continued on any its smooth compactification if and only if $F_4(Z)$ is a cusp form. Thus

$$H^{4,0}(\widetilde{\mathcal{M}}_{\Gamma_{2n+2,2d}}(L_{2n+2,2d})) \cong S_4(\Gamma_{2n+2,2d}, det).$$

Problem: To construct at least one such cusp form.

I can do this using my model of the **automorphic discriminant** of the moduli spaces of Enriques surfaces. This modular form belongs to the kernel of the hyperbolic differential operator for $U \oplus D_8(-1)$ and its determines a gravitational correction of the FHSV-model.

8. Irrationality of moduli spaces of GKVs

Main Theorem. $\mathcal{M}(Kum^n, h_{2d}^{split})$ is irrational (i.e., its Kodaira dimension is non-negative) if

$$2n+2 = a_1^2 + \cdots + a_k^2,\ \ k = 1,3,5,7,\ \ a_i \neq 0,$$
$$2d = b_1^2 + \dots b_{8-k}^2,\ \ b_j \neq 0,$$

and the following cups condition is true

$$\frac{a_1 \cdot \ldots \cdot a_k}{gcd(a_1, \ldots, a_k)} \ \text{ or } \ \frac{b_1 \cdot \ldots \cdot b_{8-k}}{gcd(b_1, \ldots, b_{8-k})} \ \text{ is even.}$$

We take the odd Jacobi theta-series $\vartheta(\tau, z)$ of characteristic 2

$$\sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8} r^{n/2} = -q^{1/8} r^{-1/2} \prod_{n\geqslant 1} (1-q^{n-1}r)(1-q^n r^{-1})(1-q^n),$$

$$q = e^{2\pi i \tau}, \quad r = e^{2\pi i z}, \quad \tau \in \mathbb{H}_1, \quad z \in \mathbb{C},$$

$$\vartheta(\tau, -z) = -\vartheta(\tau, z), \qquad \frac{\partial\, \vartheta(\tau, z)}{\partial\, z}\Big|_{z=0} = 2\pi i\, \eta(\tau)^3.$$

We take the following model of the root lattice D_8

$$D_8 = \{(a_1, \ldots, a_8) \in \mathbb{Z}^8, a_1 + \cdots + a_8 \in 2\mathbb{Z}\}.$$

Then $\vartheta(\tau, z_1) \cdot \ldots \cdot \vartheta(\tau, z_8)$ is a Jacobi form of weight 4 for the root lattice D_8. We can take its arithmetic lifting (Gr-1993)

$$\Phi_4(\tau, z_1, \ldots, z_8, \omega) = \text{G-Lift}\,(\vartheta(\tau, z_1) \cdot \ldots \cdot \vartheta(\tau, z_8)),$$

$$\Phi_4 \in M_4(O^+(2U \oplus D_8(-1)), \chi_2).$$

10. Proof: a game with sums of non-zero squares

1) We embed the diagonal quadratic form $\langle 2n+2\rangle \oplus \langle 2d\rangle$ in D_8:

$$(u_{2n+2} = (a_1, \ldots, a_k),\ v_{2d} = (b_1, \ldots b_{8-k})) \in D_8.$$

2) We get an embedding of the lattice $L_{2n+2,2d} \to 2U \oplus D_8(-1)$ and the corresponding homogeneous tube domains.

3) Using the pull-back of the modular form $\Phi_4(\tau, z_1, \ldots, z_8, \omega)$, we get a $\Gamma_{2n+2,2d}$-modular form of weight 4:

$$F_4(\tau, z_1, z_2, \omega) =$$

$$= \text{G-Lift}\left(\left(\vartheta(a_1 z_1) \cdot \ldots \cdot \vartheta(a_k z_1)\right) \cdot \left(\vartheta(b_1 z_2) \cdot \ldots \cdot \vartheta(b_{8-k} z_2)\right)\right).$$

4) $F_4 \neq 0$ if all a_i and b_j are non zero and F_4 is a cusp form if the cusp condition is satisfied.

5) $F_4(\tau, -z_1, z_2, \omega) = -F_4(\tau, z_1, z_2, \omega)$, i.e. F_4 is anti-invariant with respect to the reflection σ_{2n+2}, if k is **odd**.

Proposition.
1) $2d$ is a sum of 5 positive squares if $2d \neq 2, 4, 6, 10, 12, 18$.

2) $2d$ is a sum of 7 positive squares if $2d \neq 2, 4, 6, 8, 14, 20$.

Proof. 169 is a sum of 1, 2, 3, 4, 5, 6 and 7 positive squares.

$169 = 13^2 = 12^2 + 5^2 = 12^2 + 4^2 + 3^2 = 11^2 + 3.4^2 =$
$12^2 + 4^2 + 2.2^2 + 1 = 10^2 + 2.5^2 + 2.3^2 + 1 = 11^2 + 5^2 + 4^2 + 2^2 + 3.1.$
Then $n - 169 = a^2 + b^2 + c^2 + d^2$ and one has to check only $n < 169$.

Conjecture. $2d$ is a sum of 3 positive squares if $2d$ does not belong to

$$\{4^l \cdot m,\ \ m \equiv 7 \bmod 8\} \cup \{4^l \cdot (1, 2, 5, 10, 13, 25, 37, 58, 85, 130)\}.$$

12. Particular series of irrational moduli spaces

- Let $n = 2$. Then $2n + 2 = 6 = 4 + 1 + 1$. Therefore
$\mathcal{M}(Kum^2, h_{2d}^{split})$ is irrational if $2d \neq 2, 4, 6, 10, 12, 18$.
The same result we have for $n = 5$ (cusp), $6, 8, 10, \dots$.

- The "dual" result for h_6 (and also for $2d = 12, 14, 18, 22, \dots$):
$\mathcal{M}(Kum^n, h_6^{split})$ is irrational if $n \neq 2, 4, 5, 8$.

- Let $n = 3$. Then $2n + 2 = 8 = 2^2 + 4.1$. Therefore
$\mathcal{M}(Kum^3, h_{2d}^{split})$ is irrational if $2d = 6, 12, 14, 18, \dots$ is a sum of three positive squares.

- Let $n = 6$. Then $2n + 2 = 14 = 9 + 4 + 1 = 3.4 + 2.1$. Then
$\mathcal{M}(Kum^6, h_{2d}^{split})$ is irrational if $2d \neq 2, 4, 10$.

- The dual result: $\mathcal{M}(Kum^n, h_{14}^{split})$ is irrational if $n \neq 4$.

- Let $2d = 4 = 2^2$. Then we get
$\mathcal{M}(Kum^n, h_4^{split})$ is irrational if $n \neq 2, 3, 5, 6, 9$.