Non-perturbative Symmetries of Little Strings and Affine Quiver Algebras

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Integrable Systems and Automorphic Forms

(Lille, 15/May/2024)

Based on: 2011.06323, 1911.08172

2009.00797 (with Amer Iqbal)

1811.03387 (with Brice Bastian)

2212.09602 (with Baptiste Filoche)

2311.03858 (with Baptiste Filoche and Taro Kimura)

Little String Theories

Over the last decades string theory has provided insights into strongly coupled quantum systems

Specifically: prediction of existence of new interacting conformal field theories in dimensions $D > 4$

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- gravitation

$\xrightarrow[\text{gravity}]{\text{suitable decoupling of}}$

quantum field theory
- point-like degrees of freedom
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little string theory
- intrinsic string scale M_{string} remains

$\ll M_{\text{string}}$ effective QFT (point-like dofs)

$> M_{\text{string}}$ UV-comp. contains stringy dofs

- well defined energy momentum tensor

different approaches: [Witten 1995]
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Rich class of examples realised M-theory through branewebs

[Haghighat, Iqbal, Kozçaz, Lockhart, Vafa 2013]
[Haghighat, Kozçaz, Lockhart, Vafa 2013]
[SH, Iqbal 2013]
[Haghighat 2015]
[SH, Iqbal, Rey 2015]
[Haghighat, Murthy, Vafa, Vandoren 2015]

Parallel M5-branes

brane webs: M-branes arranged in a fashion to preserve (some amount of) supersymmetry

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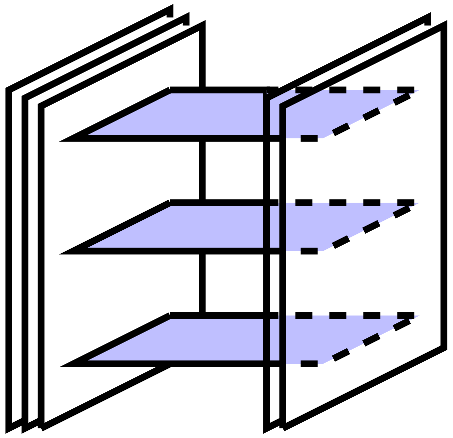
* String-like objects arise at the intersection of M5- and M2-branes



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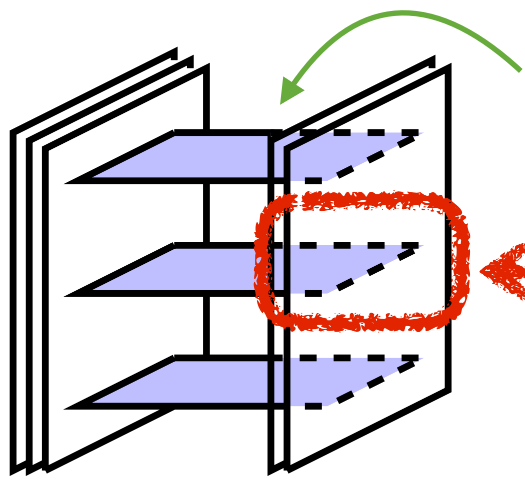

stretched M2-branes

M-string provide description of
(almost) tensionless strings in 6. dim.
relevant for $\mathcal{N}=(2,0)$ SCFT

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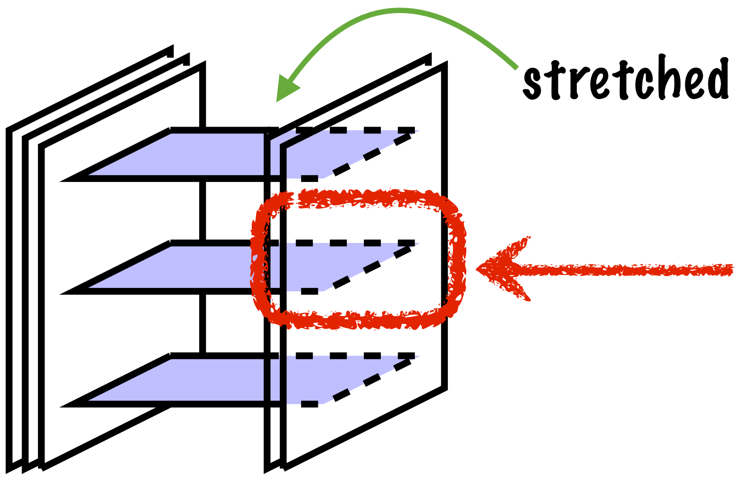

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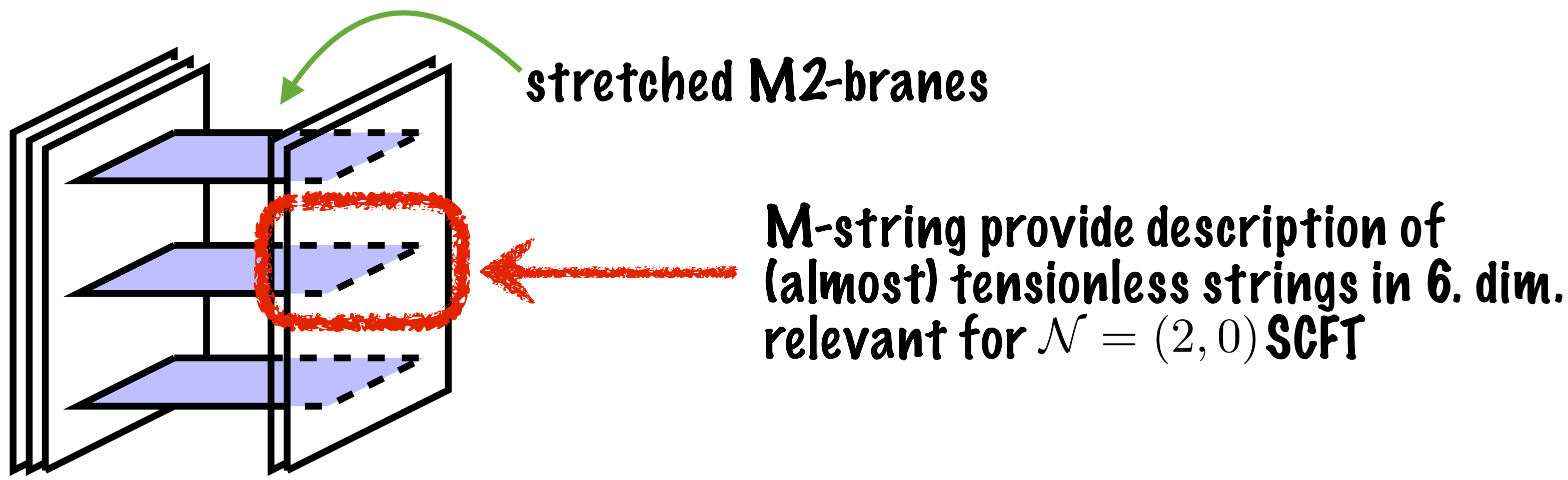

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notably: F-theory compactification on toric, non-compact Calabi-Yau threefolds

[Morrison, Vafa 1996]
[Heckman, Morrison, Vafa 2013]
[Del Zotto, Heckman, Tomasiello, Vafa 2014]
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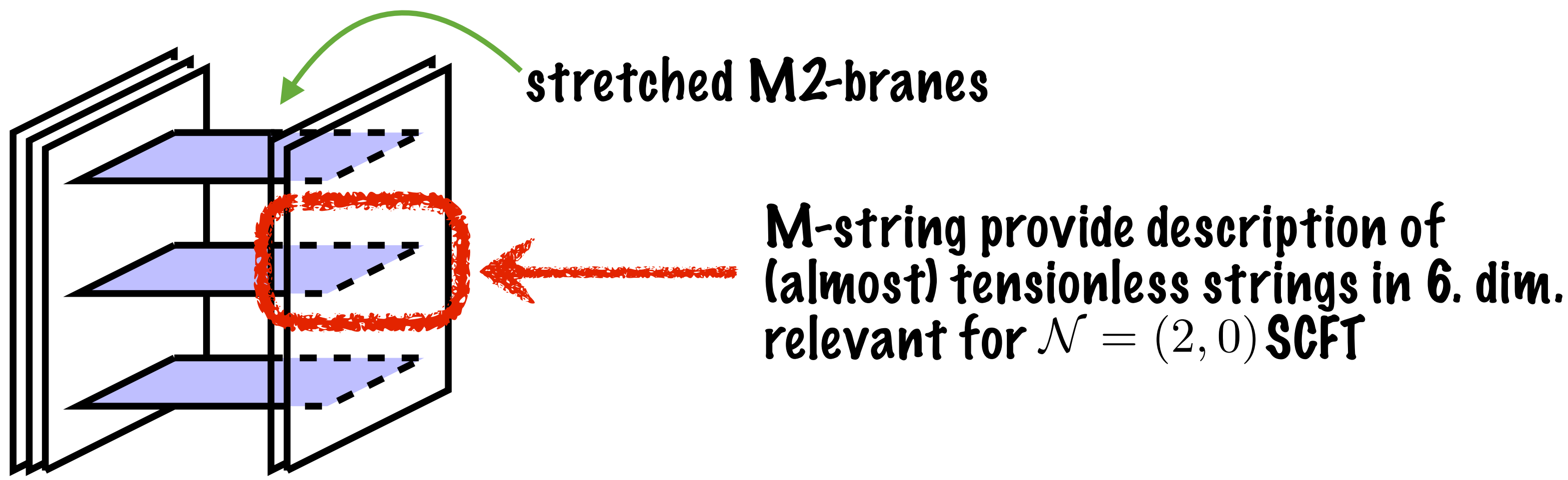

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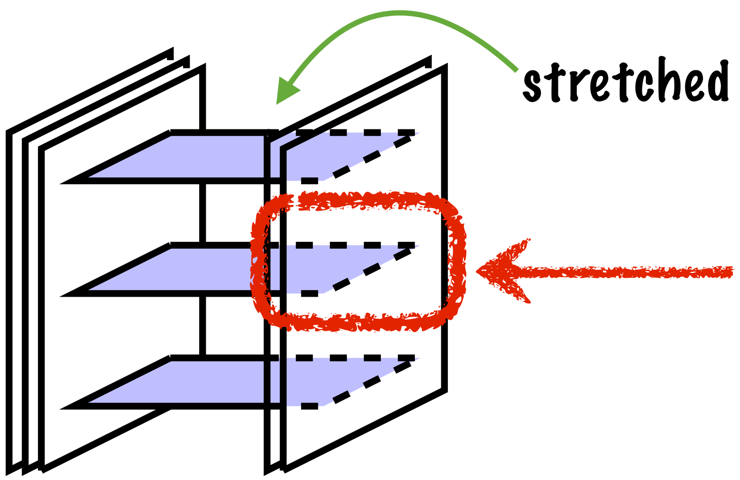

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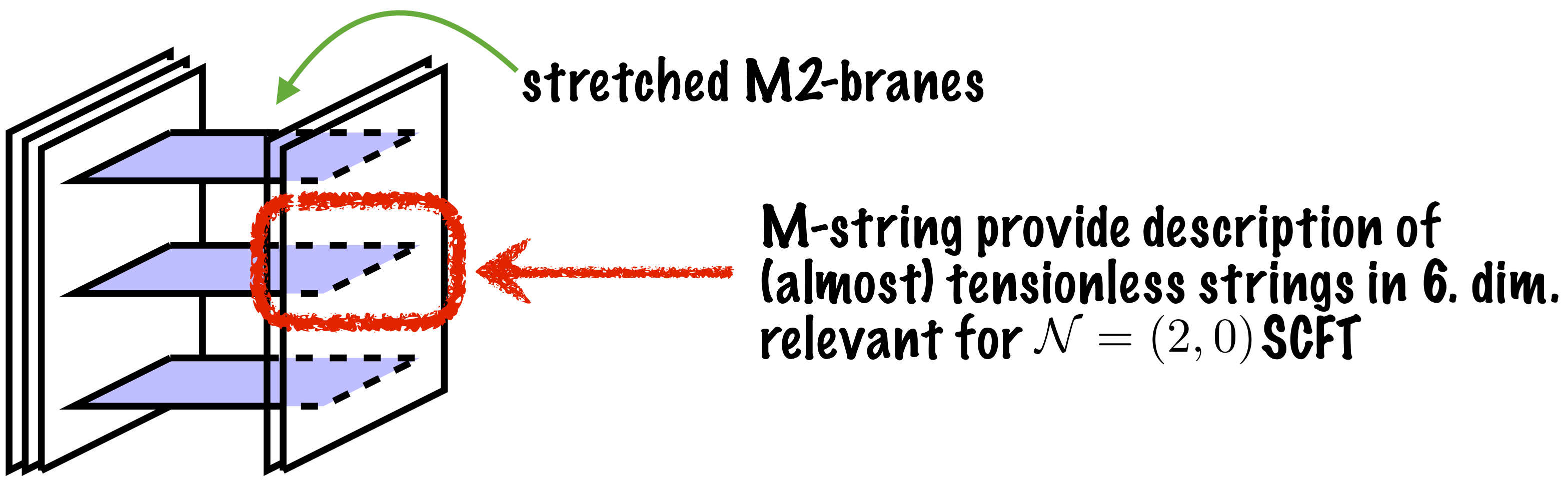

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Class of theories exhibits interesting (and non-expected) dualities (trialities)! [Bastian, SH, Iqbal, Rey 2016, 2017, 2018] [Bastian, SH 2018]

Details of the Brane Configurations

	0	1	2	3	4	5	6	7	8	9	10
M5-branes	●	●	●	●	●	●					
M2-branes	●	●					●				

$\underbrace{2\ 3\ 4\ 5}_{\mathbb{R}^4_{||}}$ $\qquad$ $\underbrace{7\ 8\ 9\ 10}_{\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M}$

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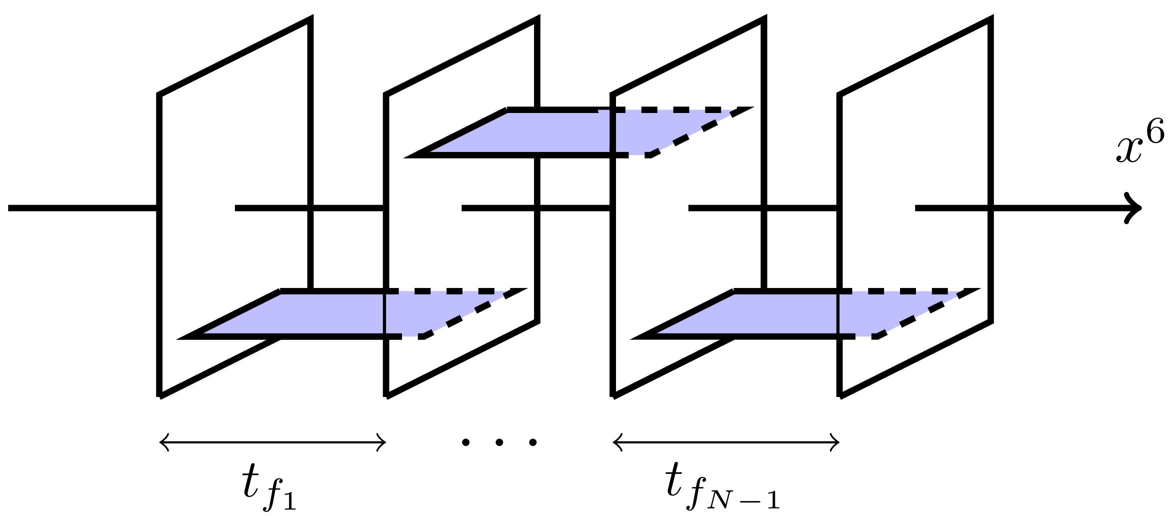
non-compact case: $\mathbb{R}$



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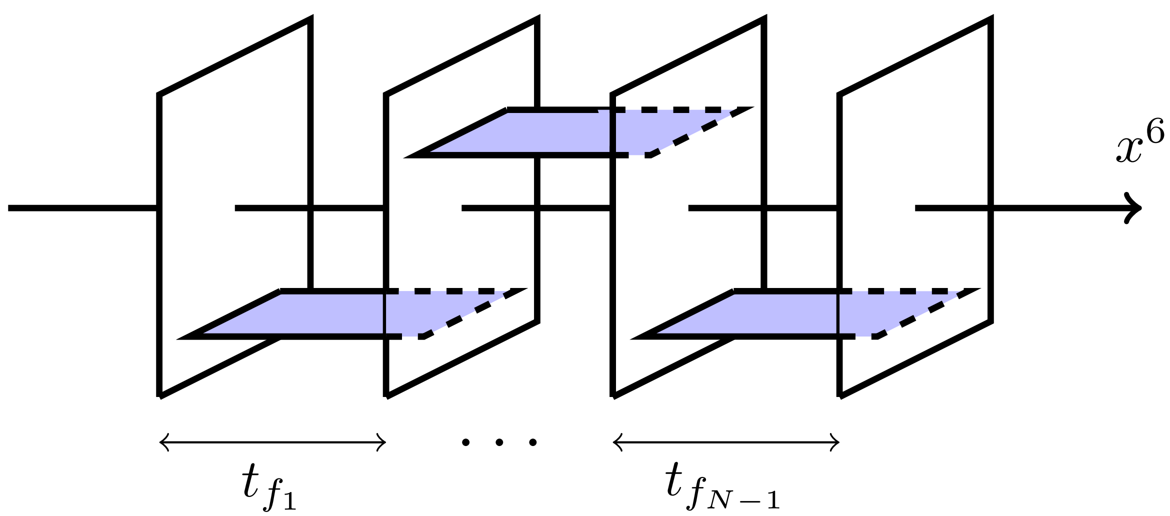

leads to CFT on M5-brane world-volume

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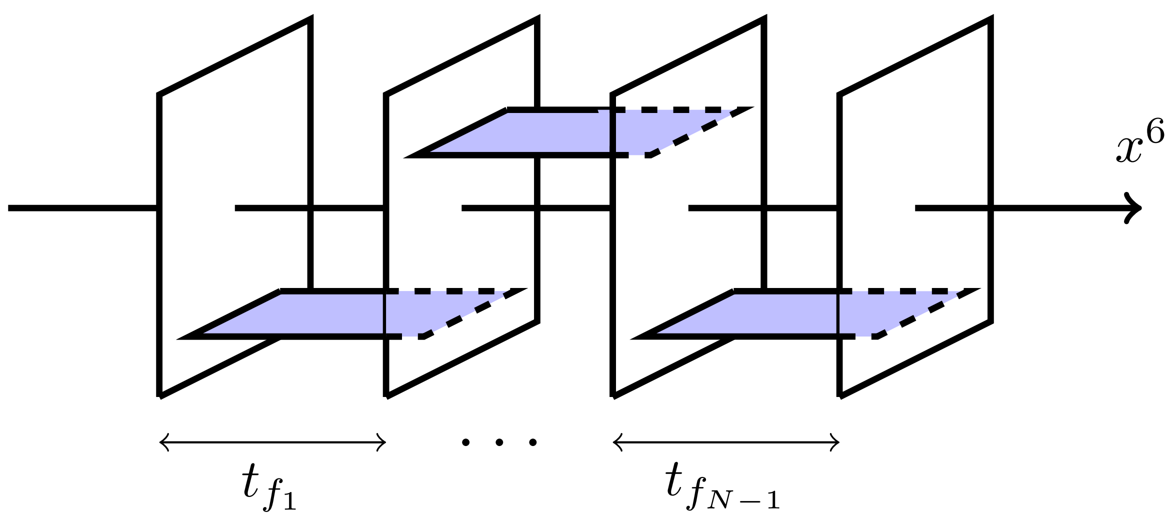
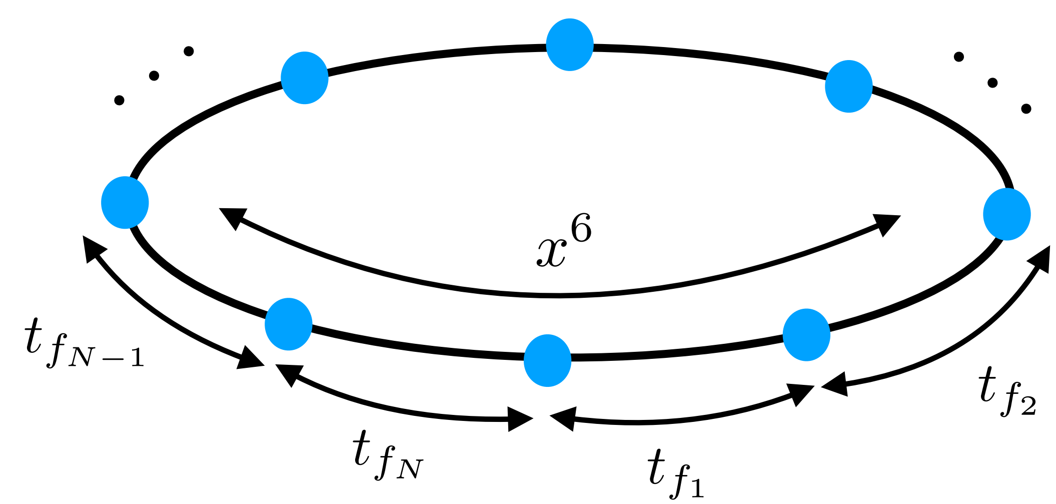
compact case: S^1 **with radius** $R_6 = \dfrac{\rho}{2\pi i}$





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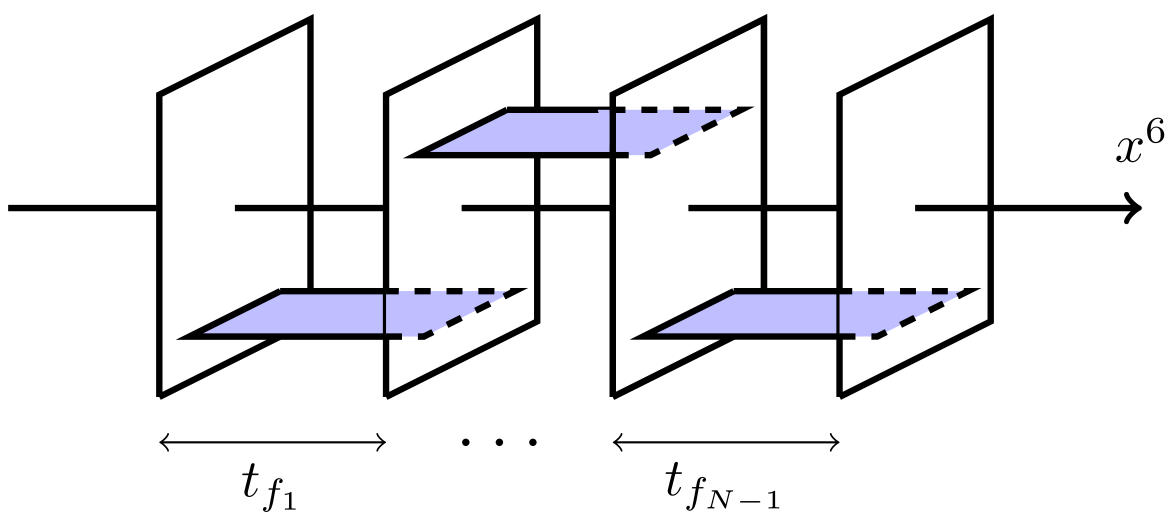

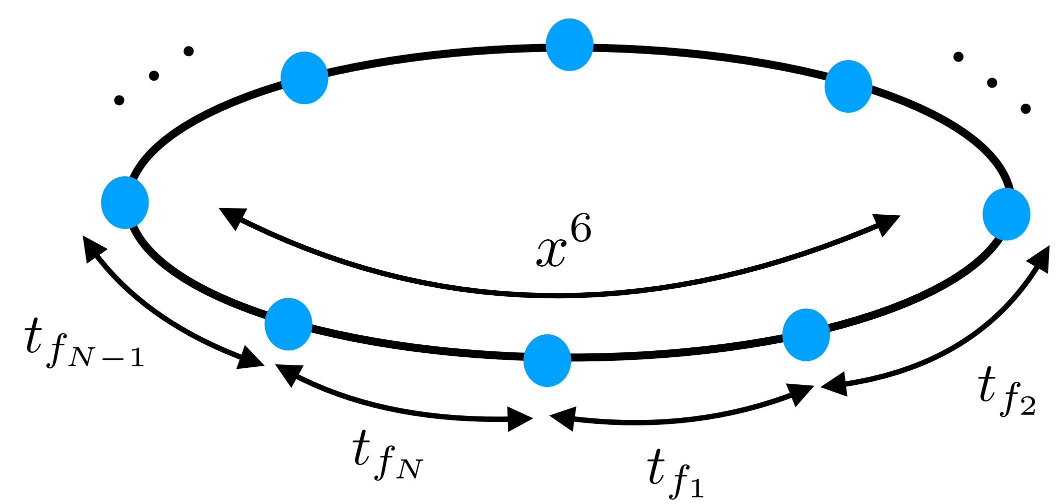

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M5-branes	●	●	●	●	●	●					
M2-branes	●	●					●				

Directions 2–5: $\mathbb{R}^4_{||}$ Directions 7–10: $\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M$

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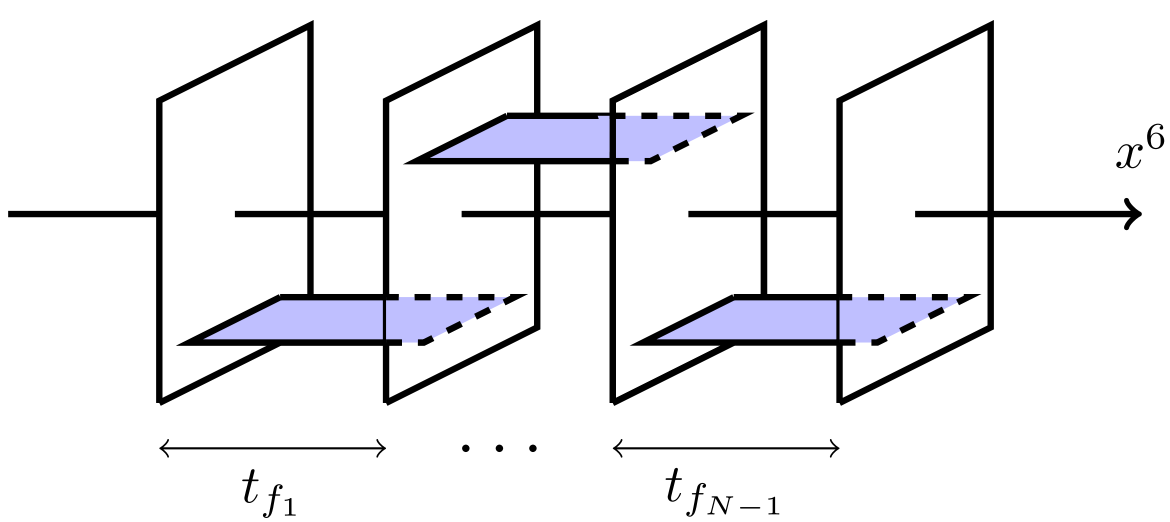

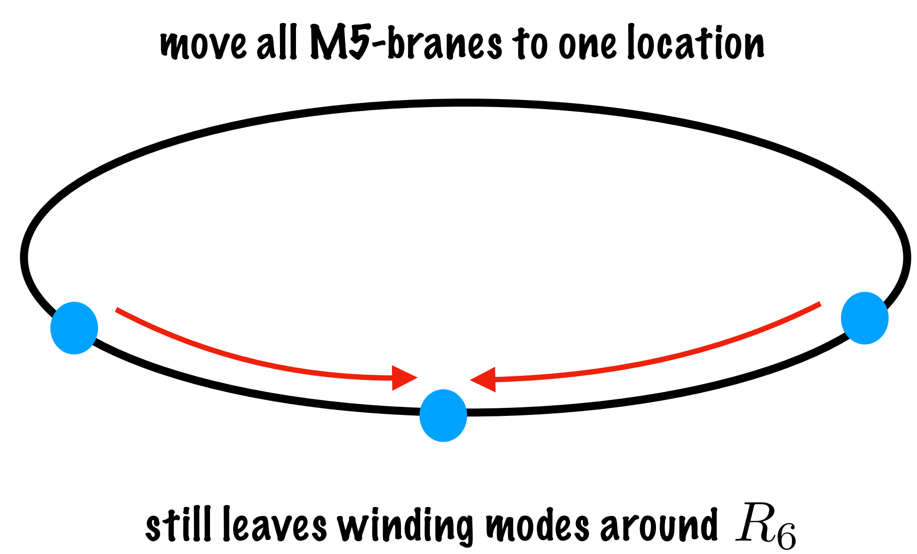

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	(0)	(1)	2	3	4	5	6	7	8	9	10
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$$\underbrace{(0)\ (1)}_{T^2 \sim S^1 \times S^1} \quad \underbrace{2\ 3\ 4\ 5}_{\mathbb{R}^4_{||}} \quad 6 \quad \underbrace{7\ 8\ 9\ 10}_{\mathrm{ALE}_{A_{M-1}} \sim \mathbb{R}^4/\mathbb{Z}_M}$$

Compactification: Compactify (0,1) to $T^2 \sim S^1 \times S^1$ with radii R_0 and $R_1 =: \dfrac{\tau}{2\pi i}$

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ϵ_2					○	○		○	○	○	○

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Deformations: there are two types of deformations with respect to the compactified (0,1)-directions introducing complex coordinates $(z_1, z_2) = (x_2 + ix_3, x_4 + ix_5)$ and $(w_1, w_2) = (x_7 + ix_8, x_9 + ix_{10})$

(0)-direct.: $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} : (z_1, z_2) \to (e^{2\pi i\epsilon_1} z_1, e^{2\pi i\epsilon_2} z_2)$ and $(w_1, w_2) \to (e^{-i\pi(\epsilon_1+\epsilon_2)} w_1, e^{-i\pi(\epsilon_1+\epsilon_2)} w_2)$

(1)-direct.: $U(1)_m : \ (w_1, w_2) \to (e^{2\pi i m} w_1, e^{-2\pi i m} w_2)$

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gauge theory: Omega-background [Nekrasov 2012]

mass-deformation

Dual Type II Setup

For vanishing mass deformation ($m = 0$) the M-brane configuration is dual to D5-NS5-branes in IIB

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NS5 branes	•	•	•	•	•	–	•			

$$\underbrace{0\ 1\ 2\ 3\ 4}_{\text{gauge theory}}\quad \underbrace{5\ 6}_{(p,q)-\text{plane}}\quad \underbrace{7\ 8\ 9}_{\text{transverse } \mathbb{R}^3}$$

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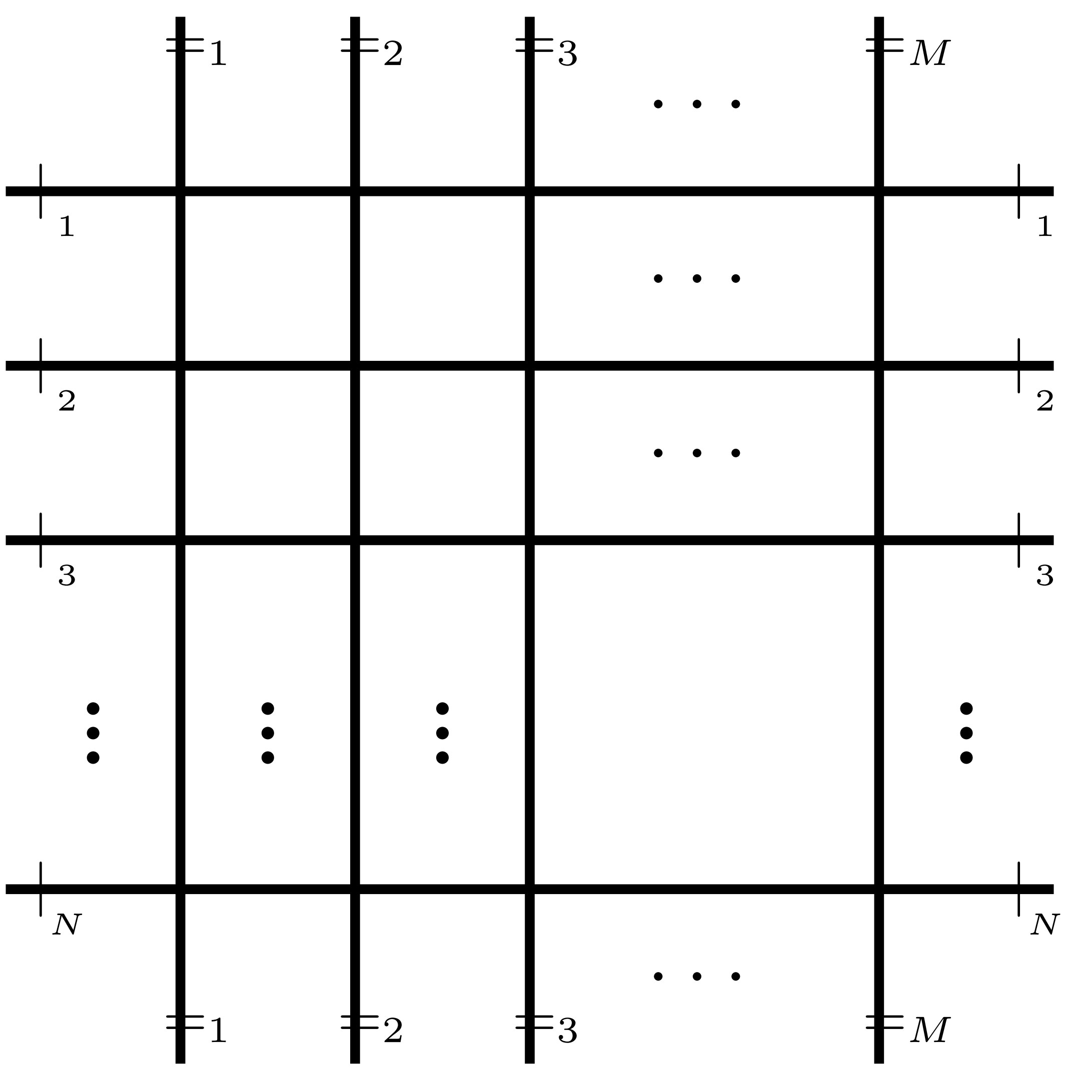

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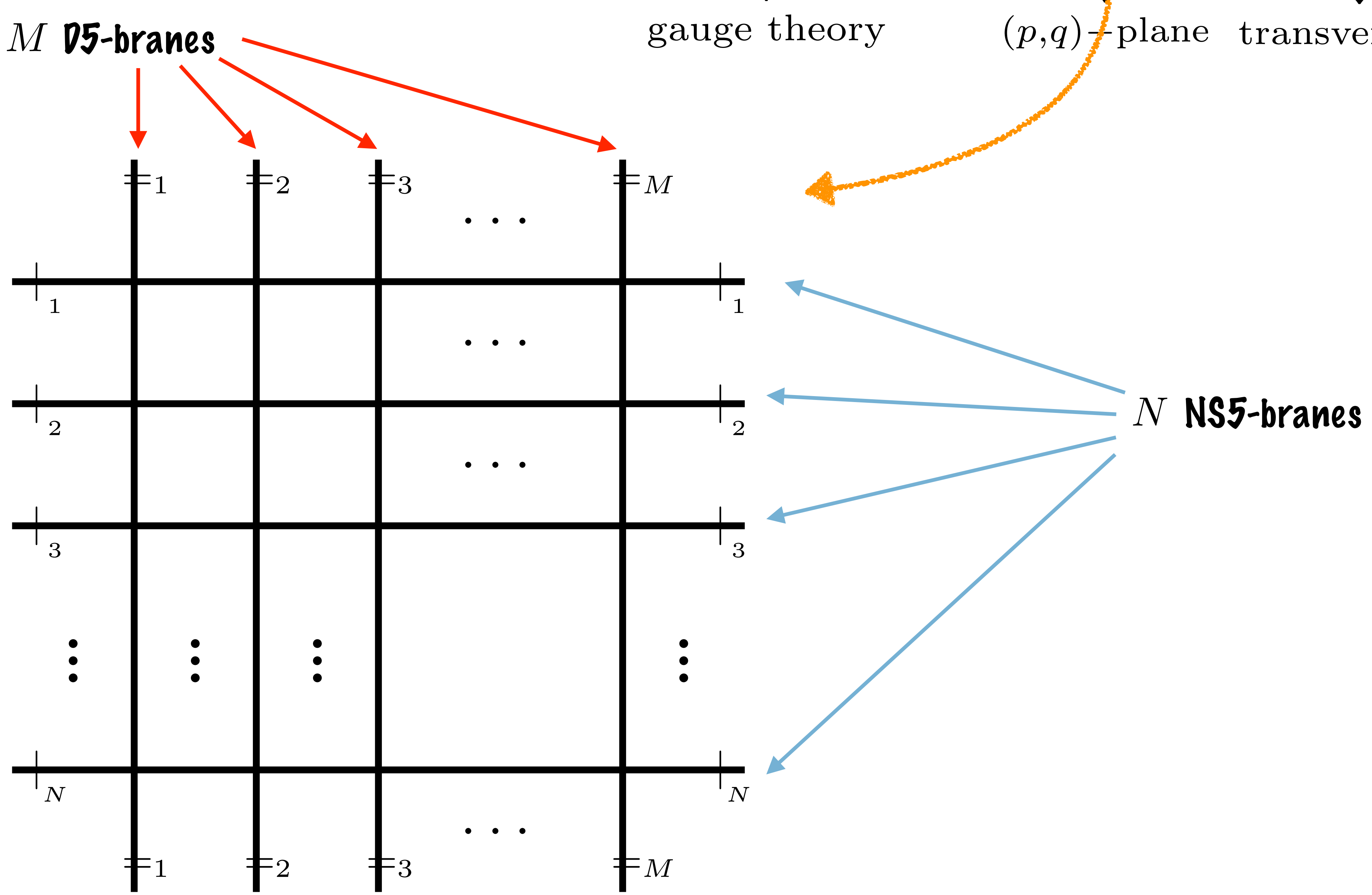

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D5 branes	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$-$			
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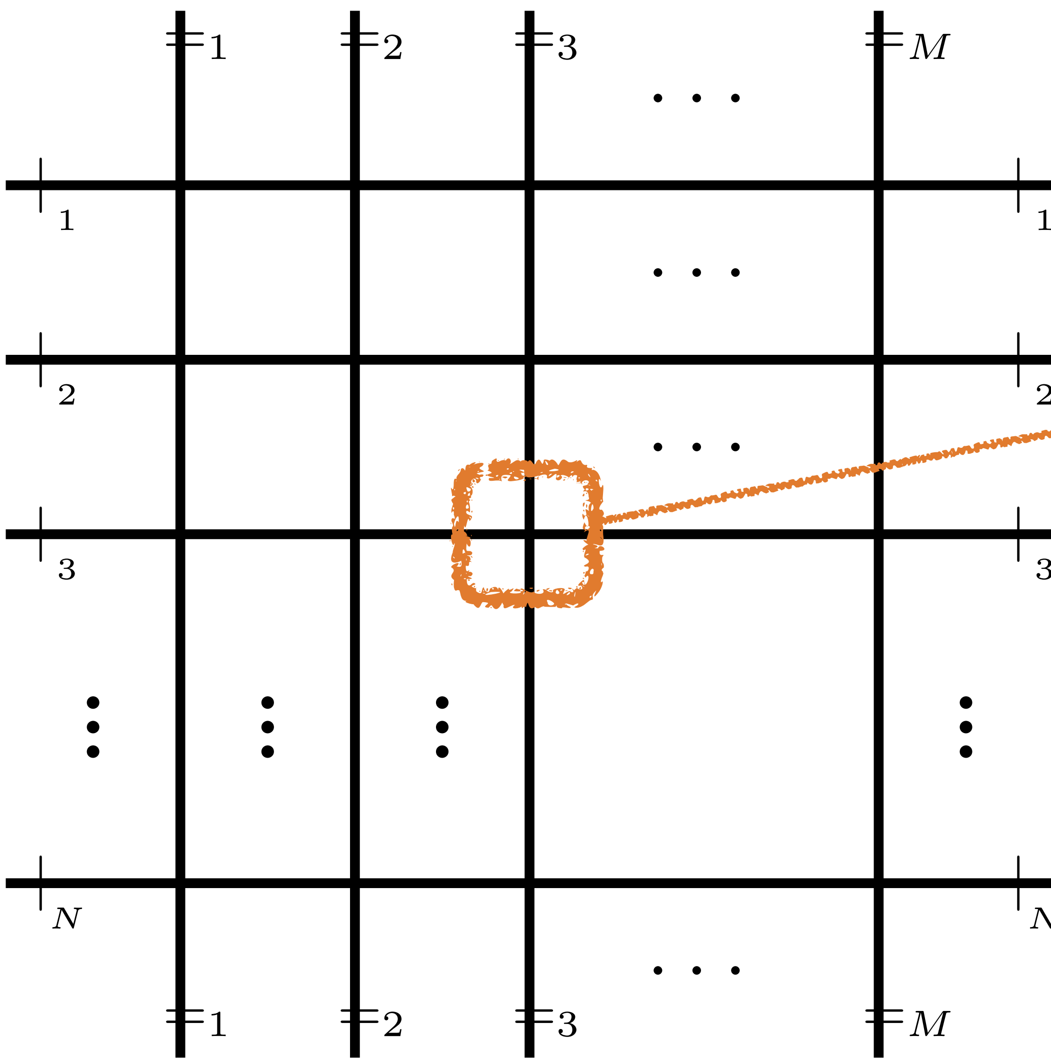
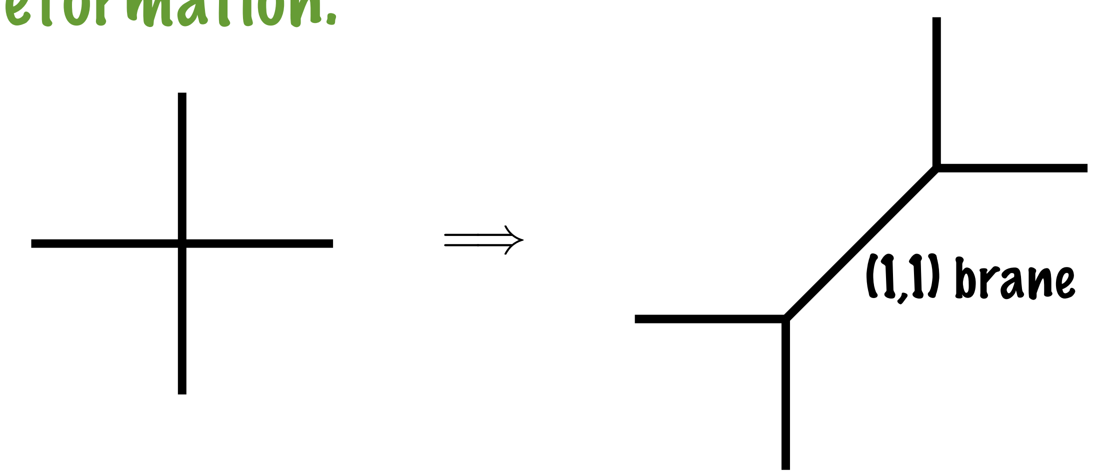


Deformation:



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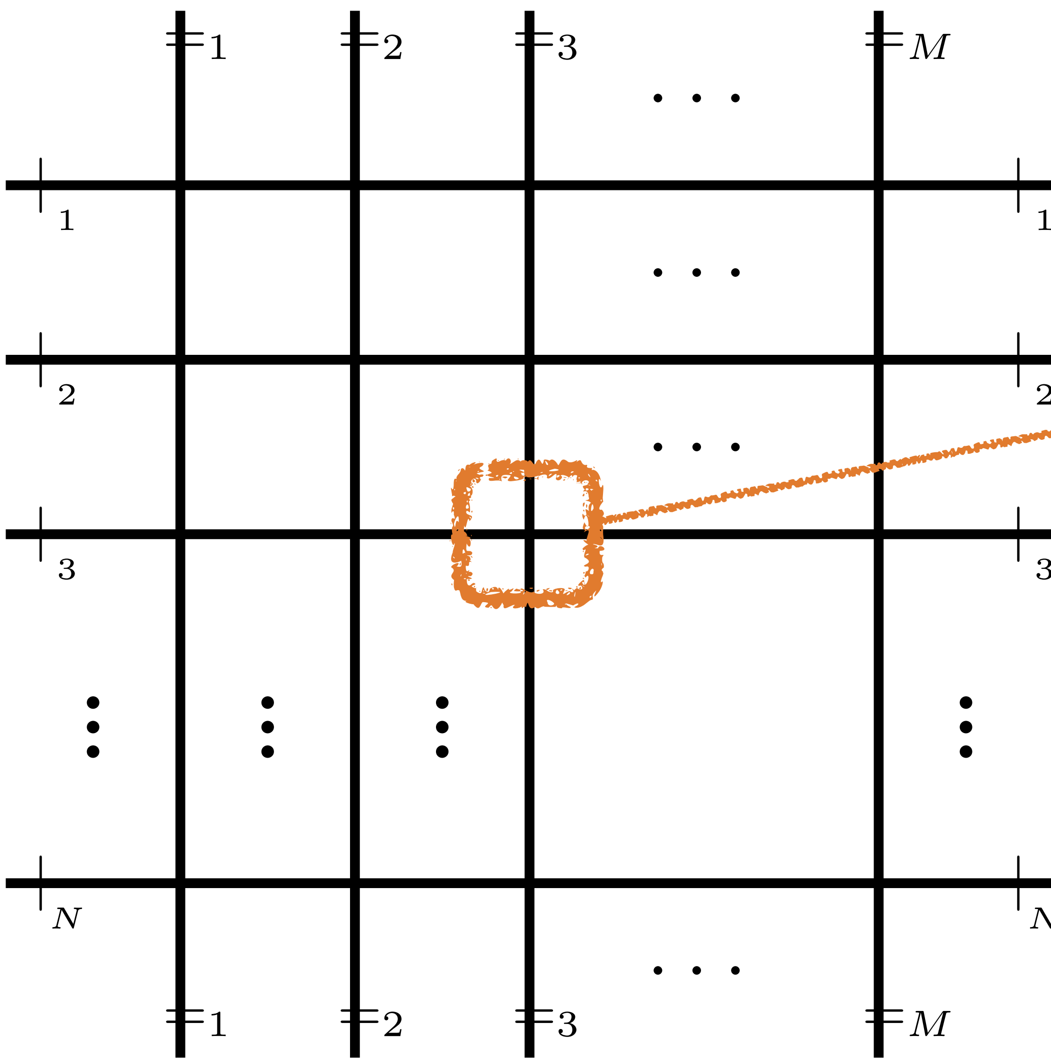

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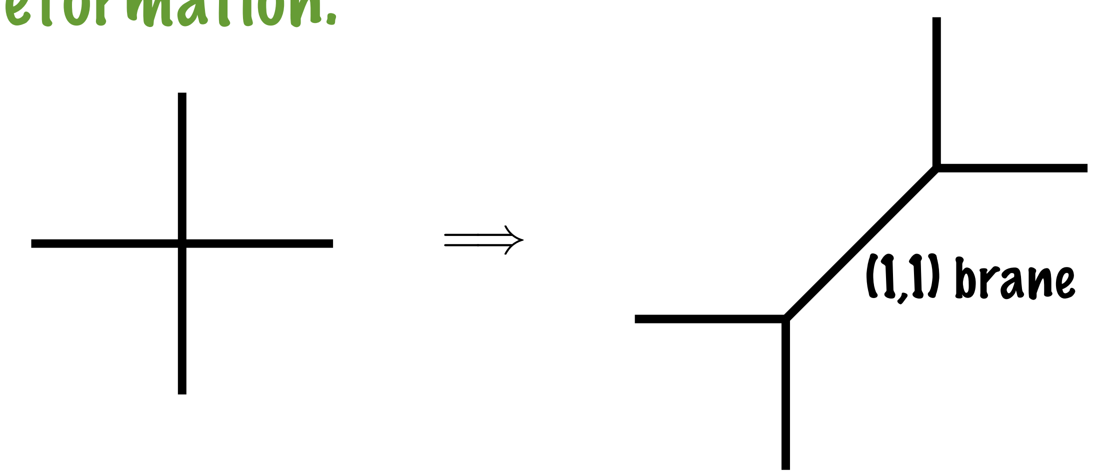

uplift the deformed type II configuration to M-th.
on an elliptically fibered Calabi-Yau threefold $X_{N,M}$

[Leung, Vafa 1997]

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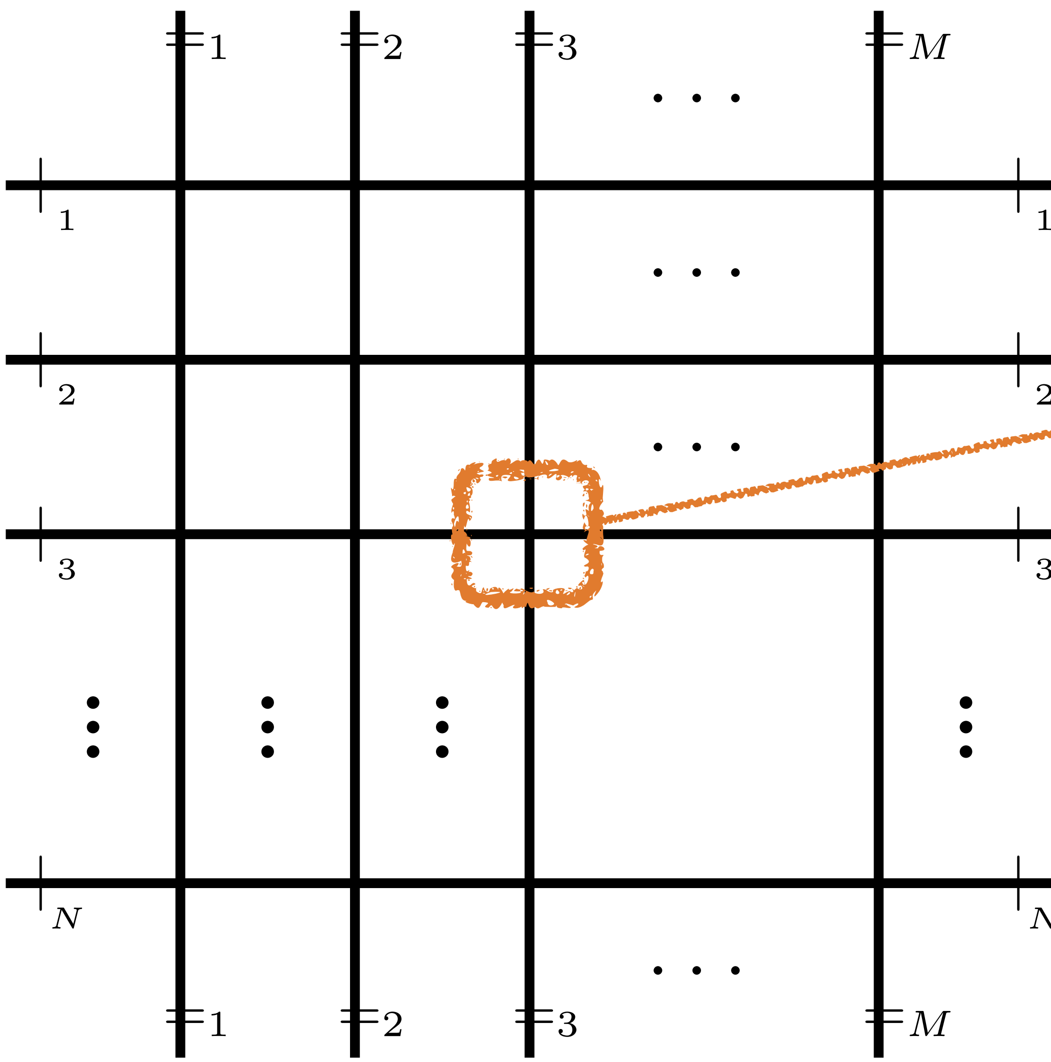

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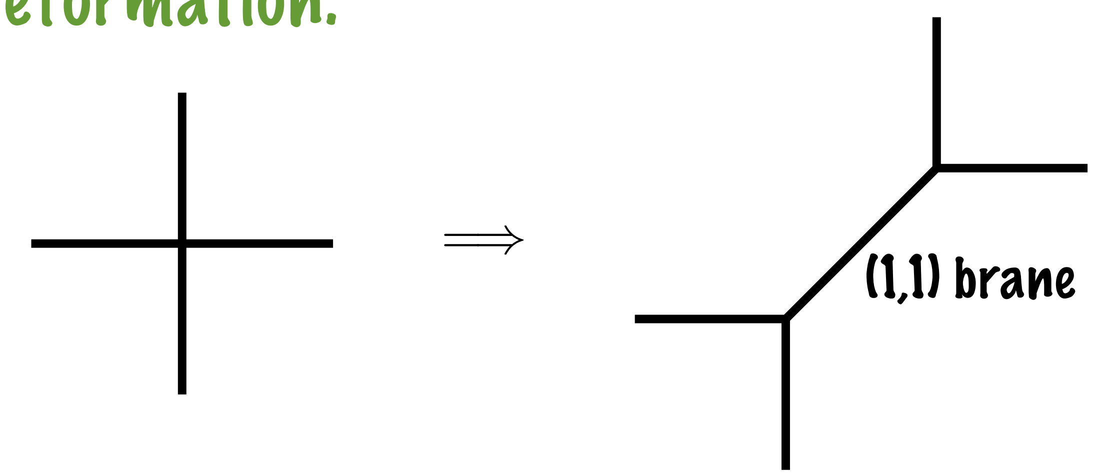

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[Leung, Vafa 1997]

topic diagram of $X_{N,M}$ same as deformed brane web

Dual Calabi-Yau 3-fold Description

2-parameter series of toric, double elliptically fibered Calabi-Yau threefolds $X_{N,M}$

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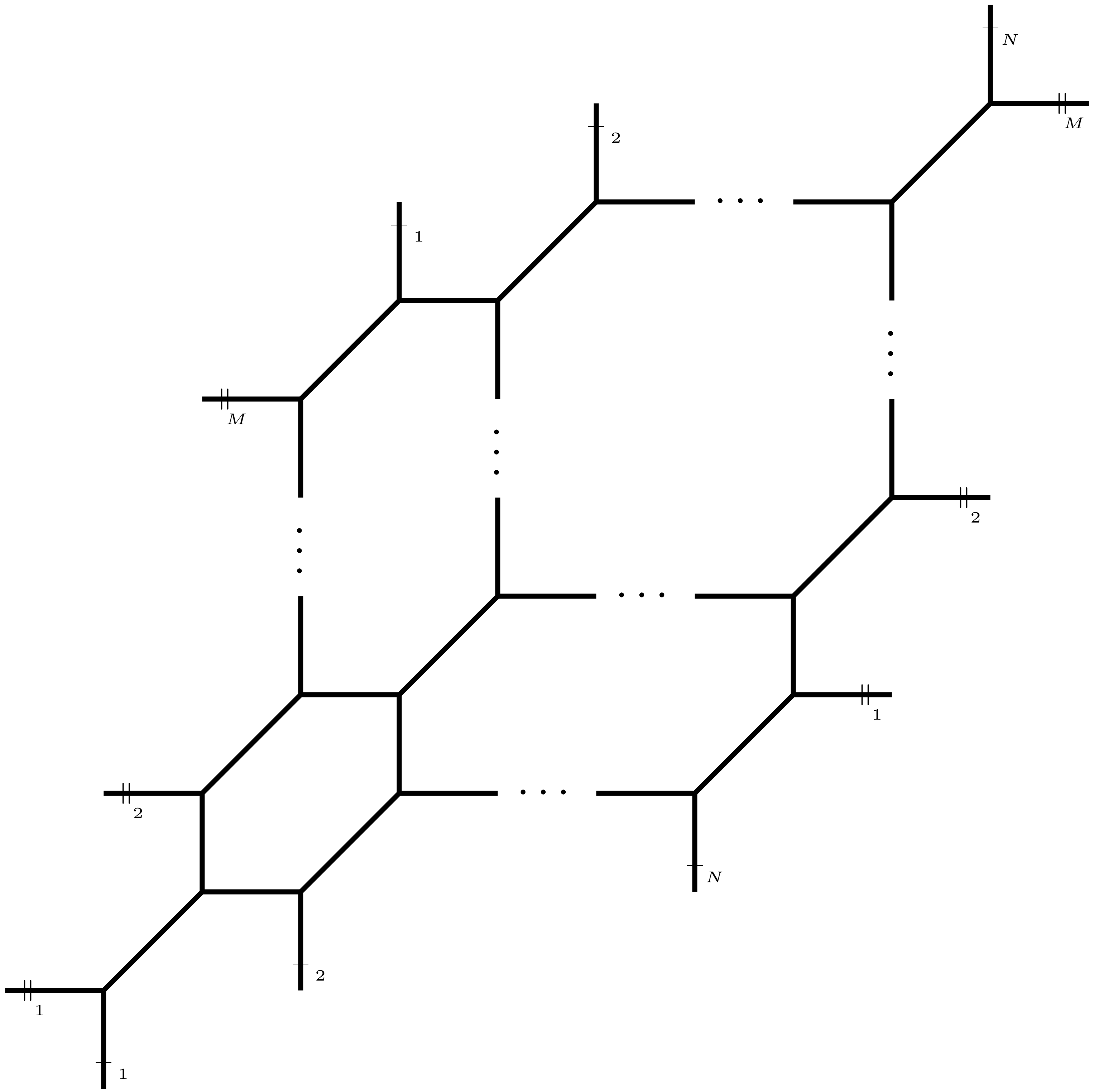
Toric Web Diagram:

- (N, M) web on a torus



Dual Calabi-Yau 3-fold Description

2-parameter series of toric, double elliptically fibered Calabi-Yau threefolds $X_{N,M}$

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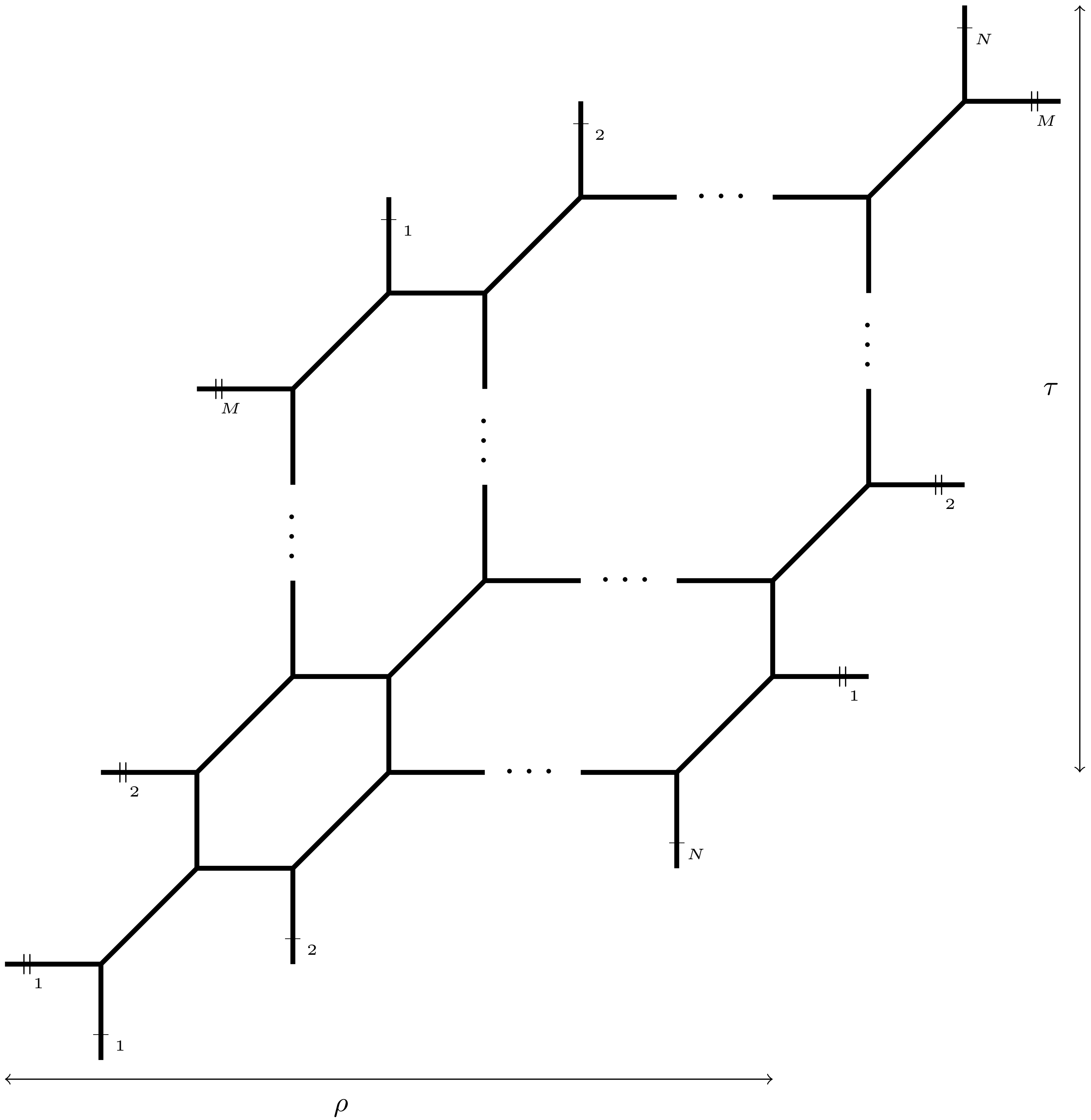

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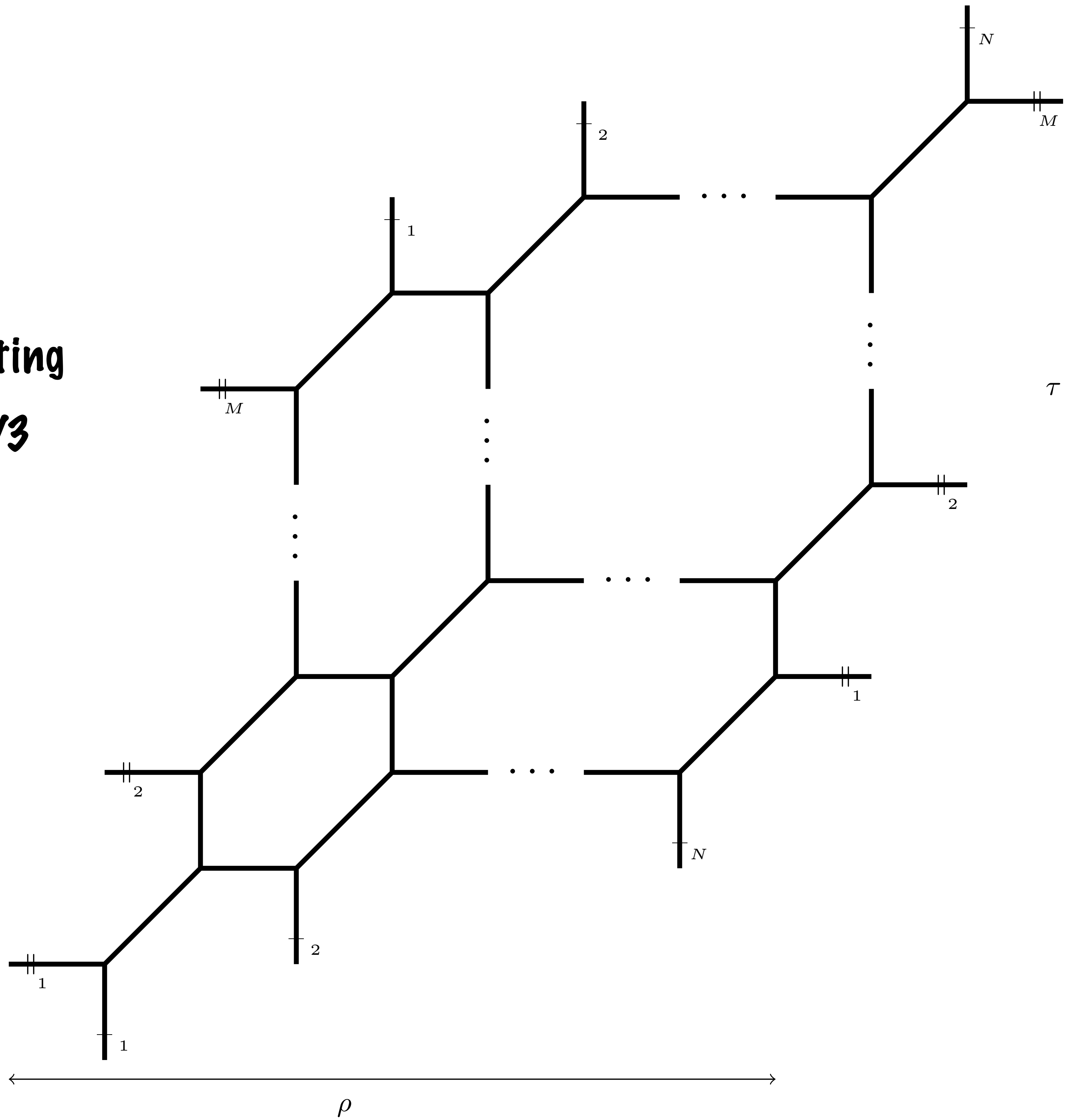
* $3NM$ different parameters representing the area of various curves C of the CY3

$$d = \int_C \omega$$



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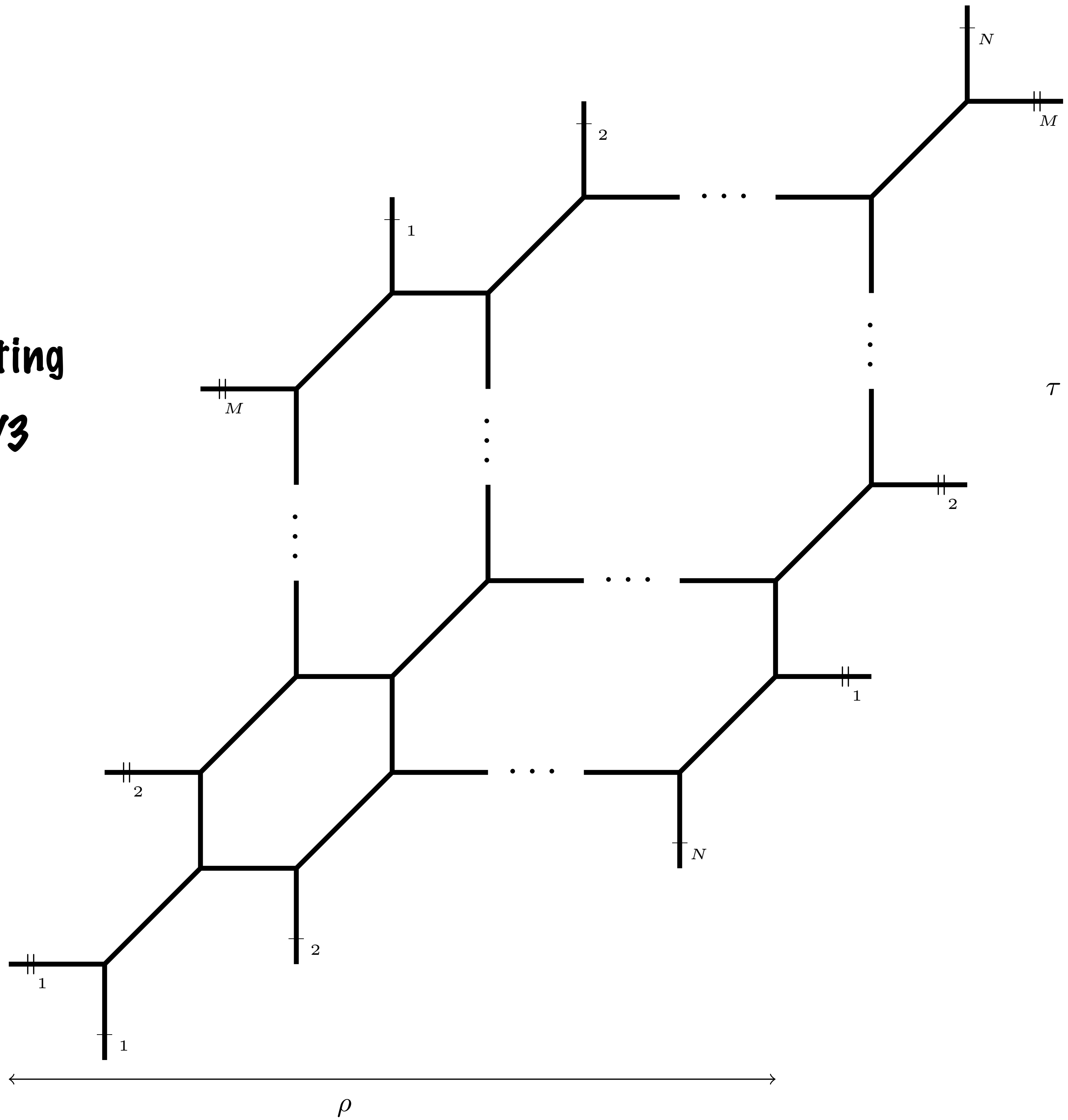

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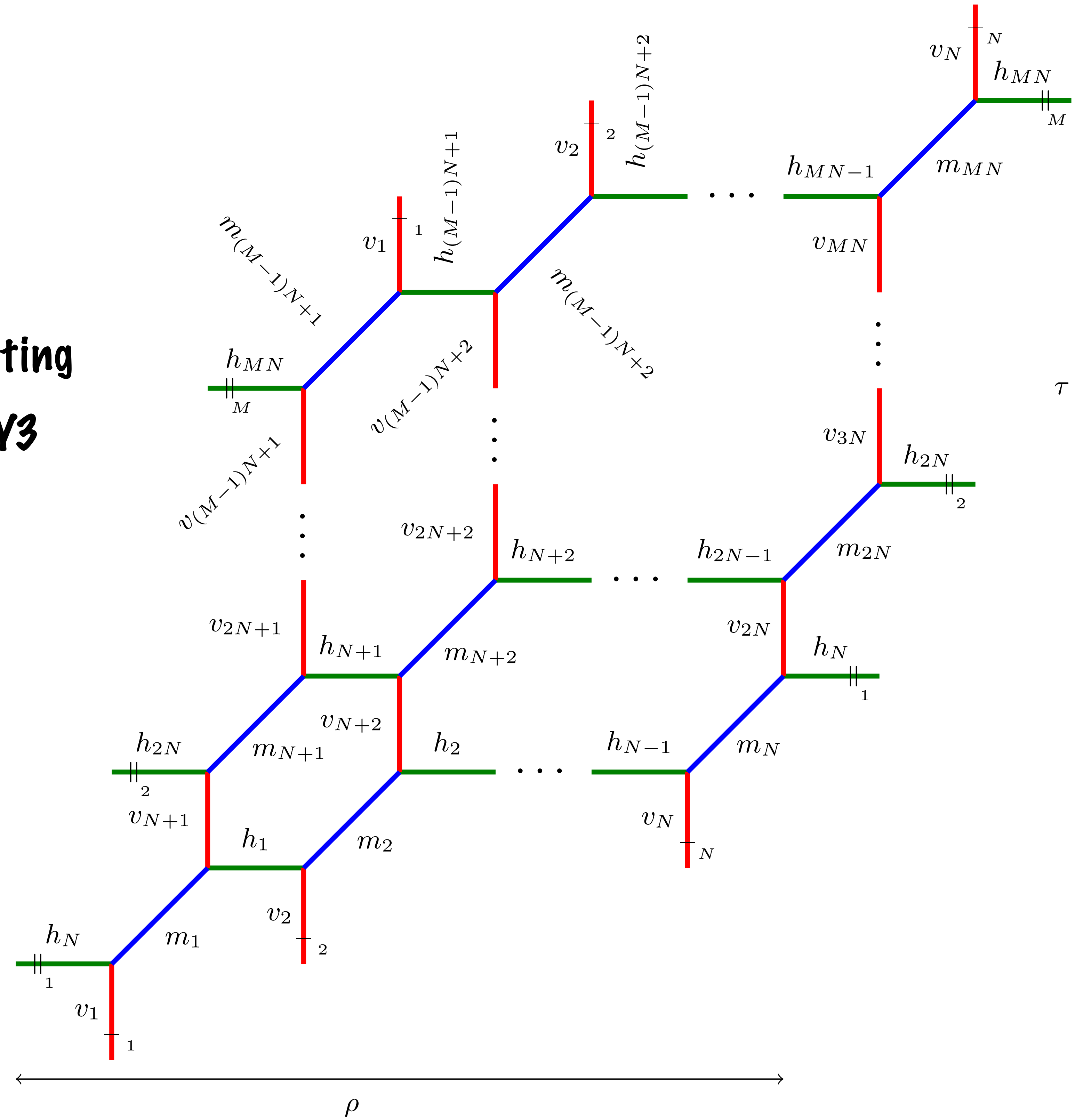

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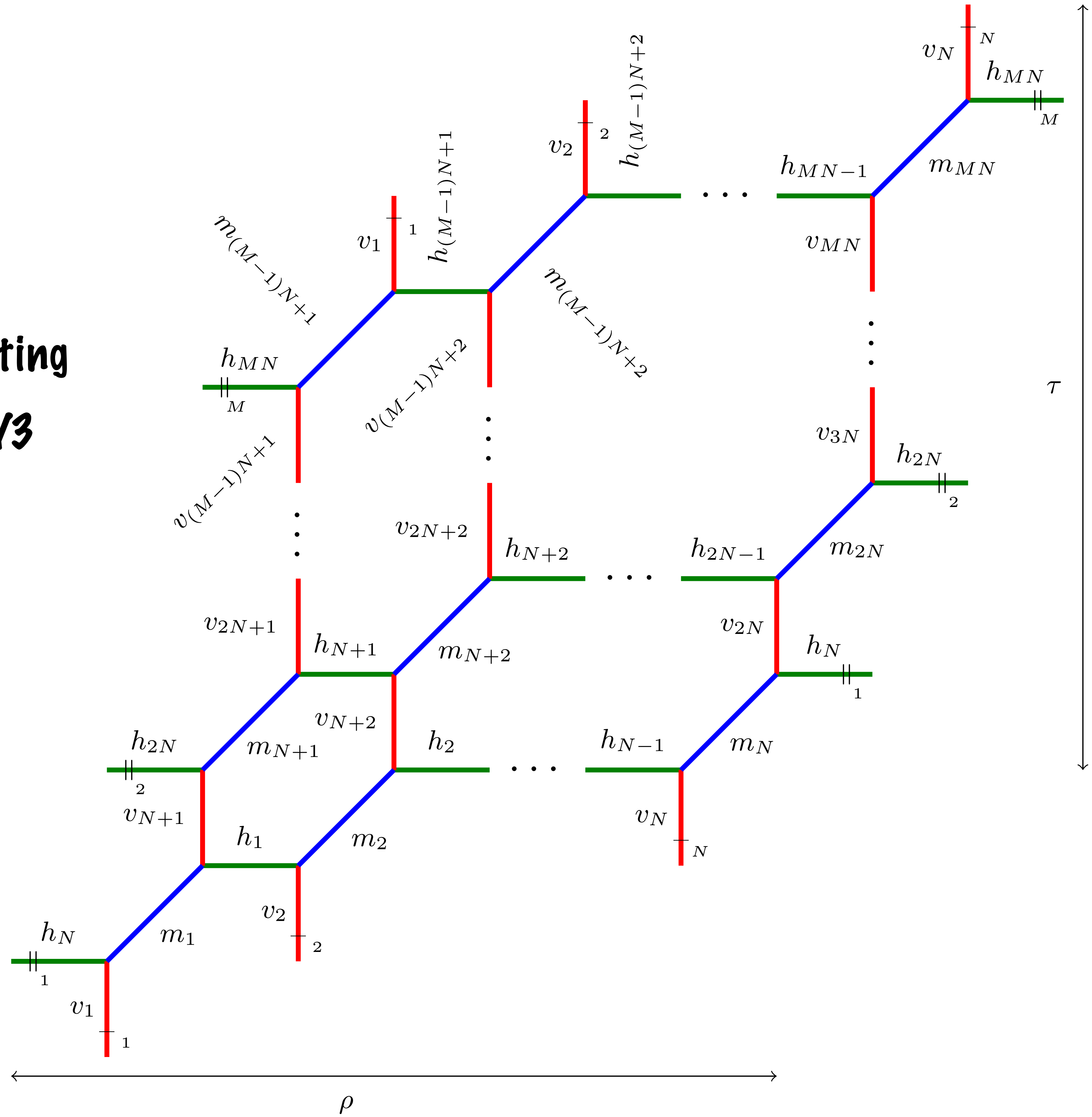

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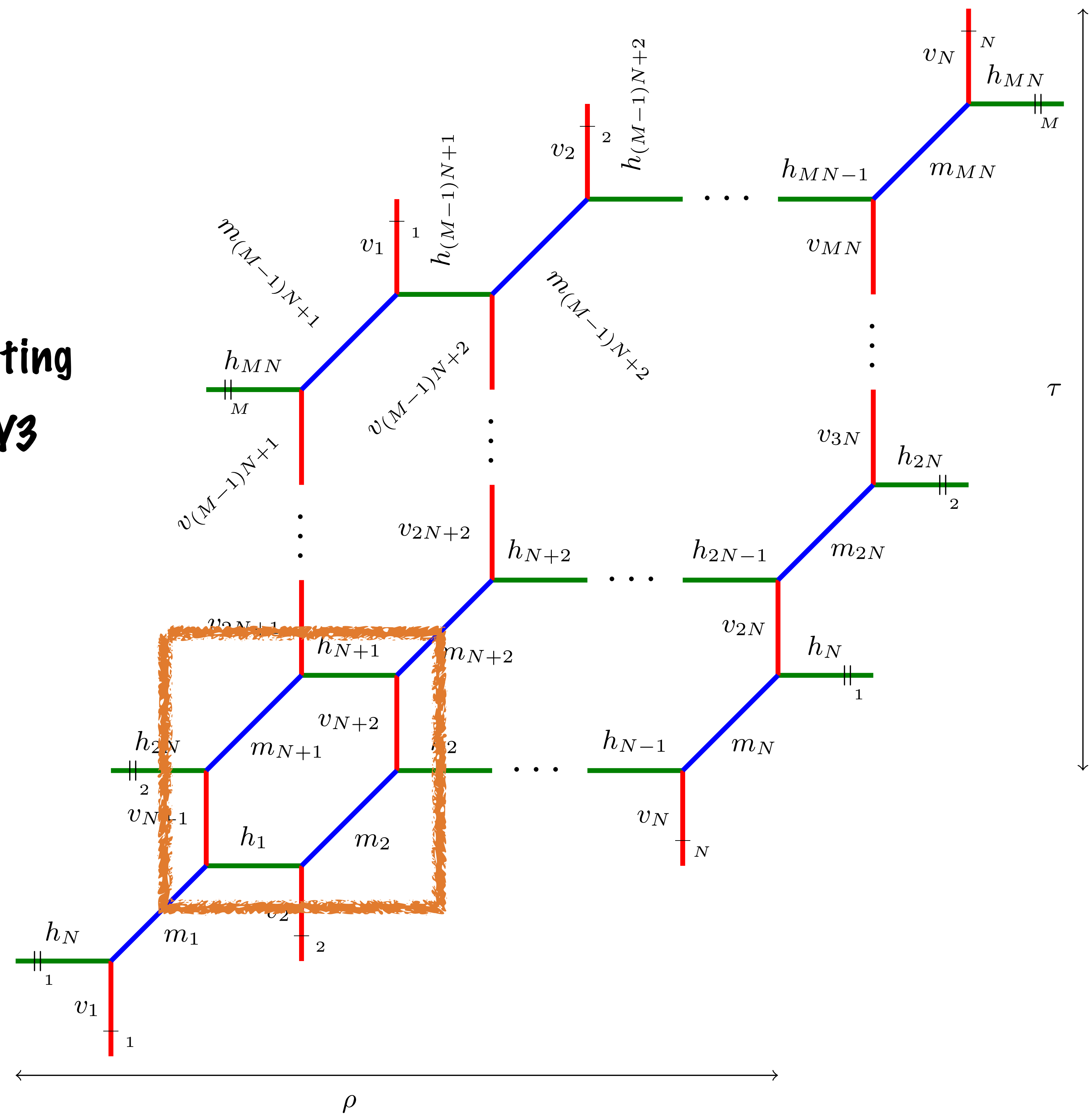

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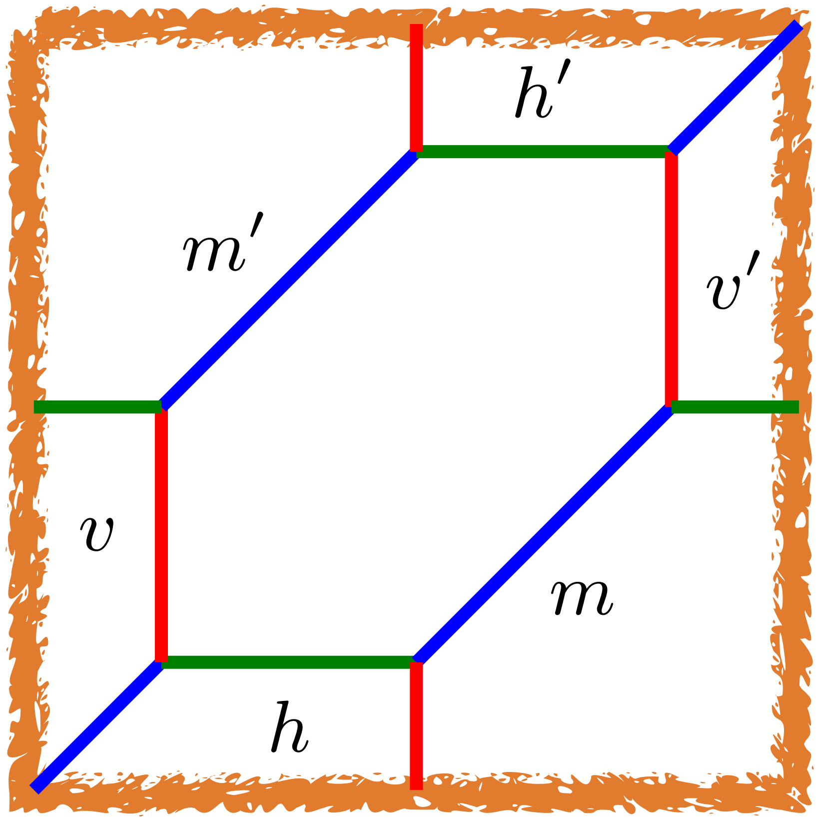

$$h + m = h' + m'$$
$$v + m' = m + v'$$

different possible choices for set of independent parameters

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Free Energy: Counts number of BPS configurations, i.e. M2-branes wrapping holomorphic curves on the CY3 $X_{N,M}$.

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Compute the topological string partition function $\mathcal{Z}_{N,M}$ using the refined topological vertex

[Aganagic, Klemm, Marino, Vafa 2003]
[Iqbal, Kozçaz, Vafa 2007]

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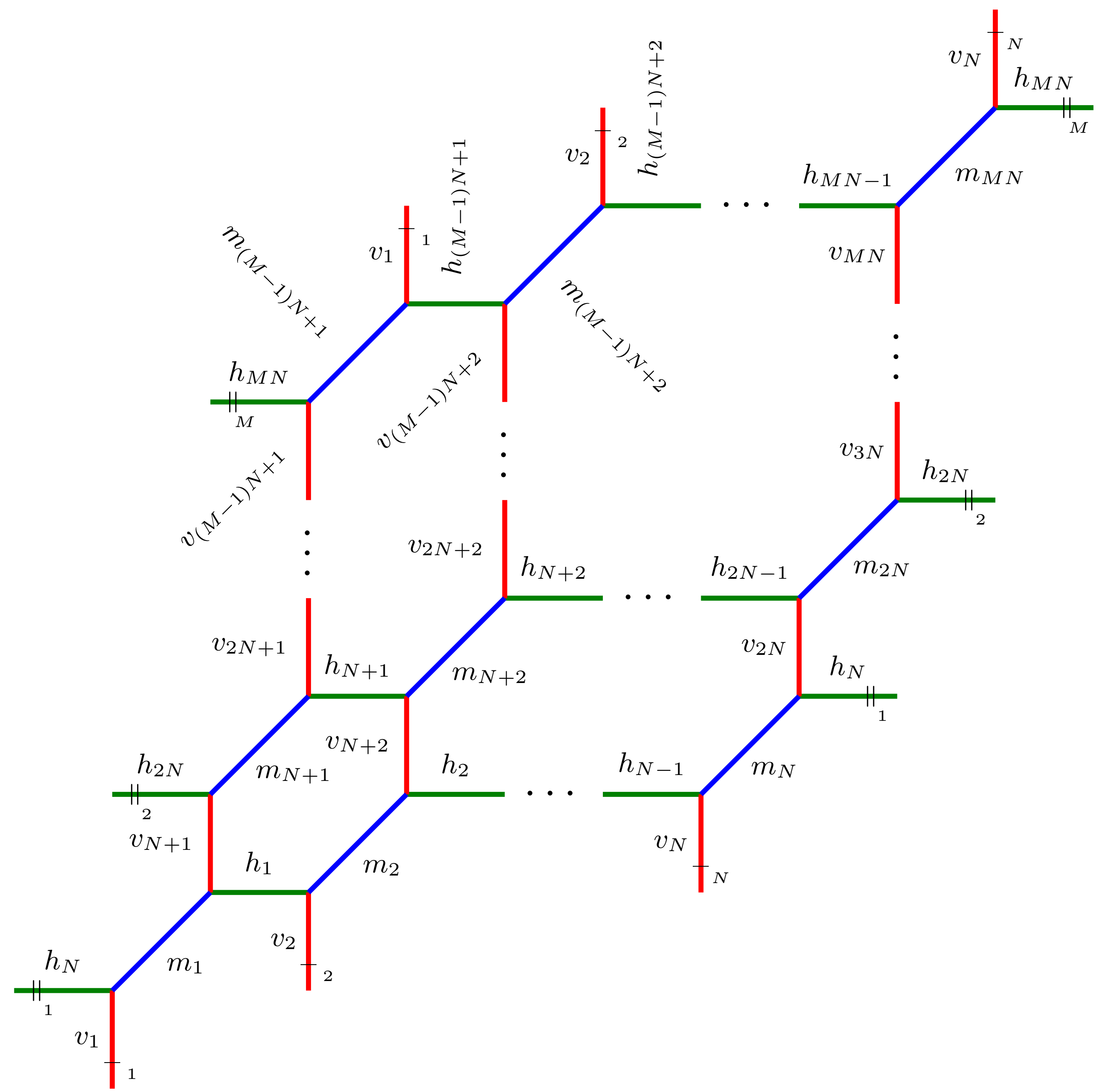

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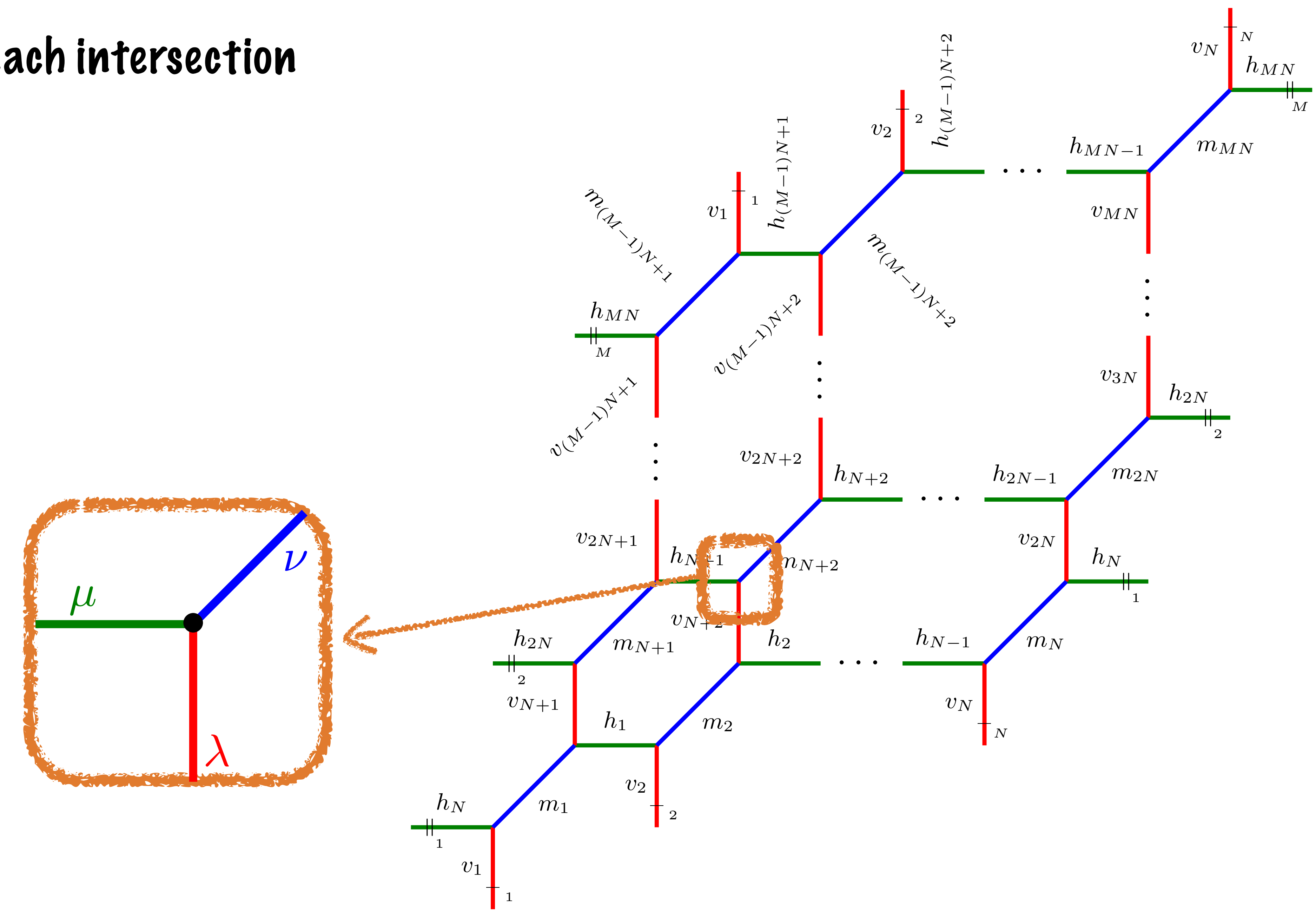
-) assign trivalent vertex to each intersection



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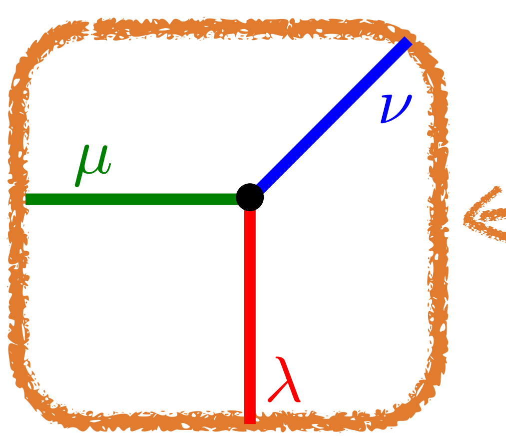

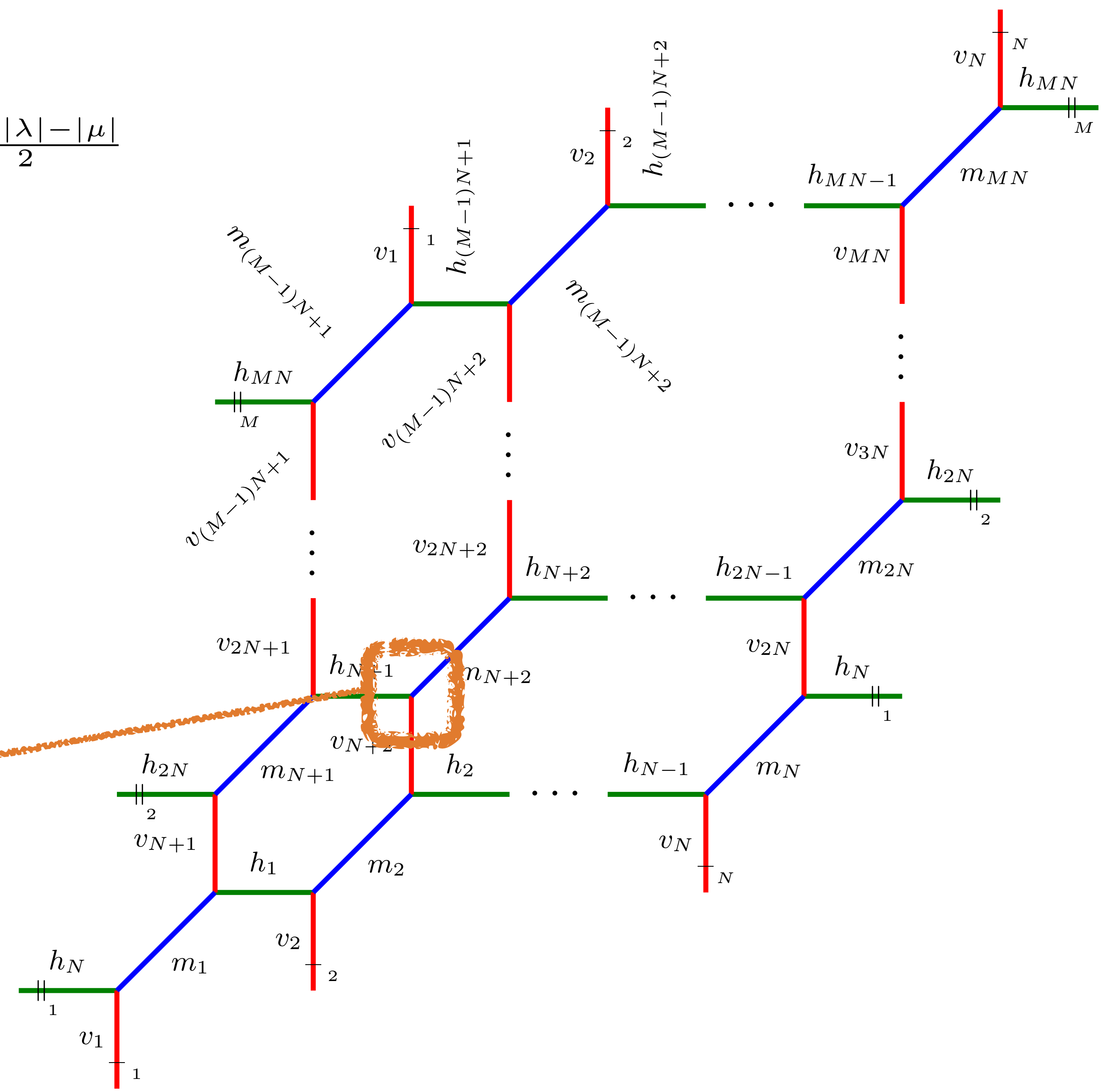

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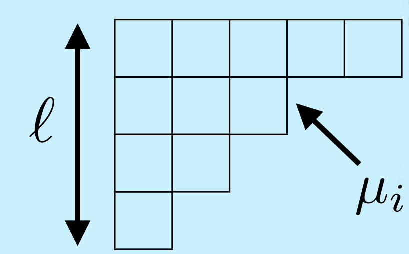
Notation:

$q = e^{2\pi i\epsilon_1}$ and $t = e^{-2\pi i \epsilon_2}$

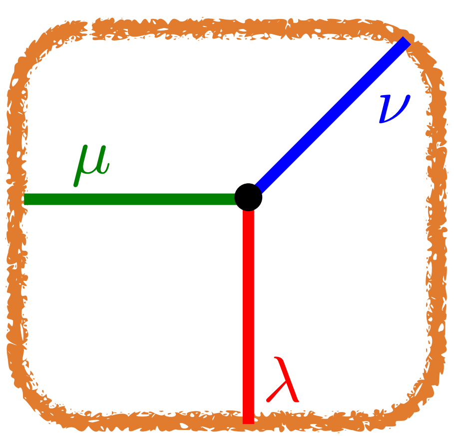
$\mu\,,\nu\,,\lambda$ integer partitions

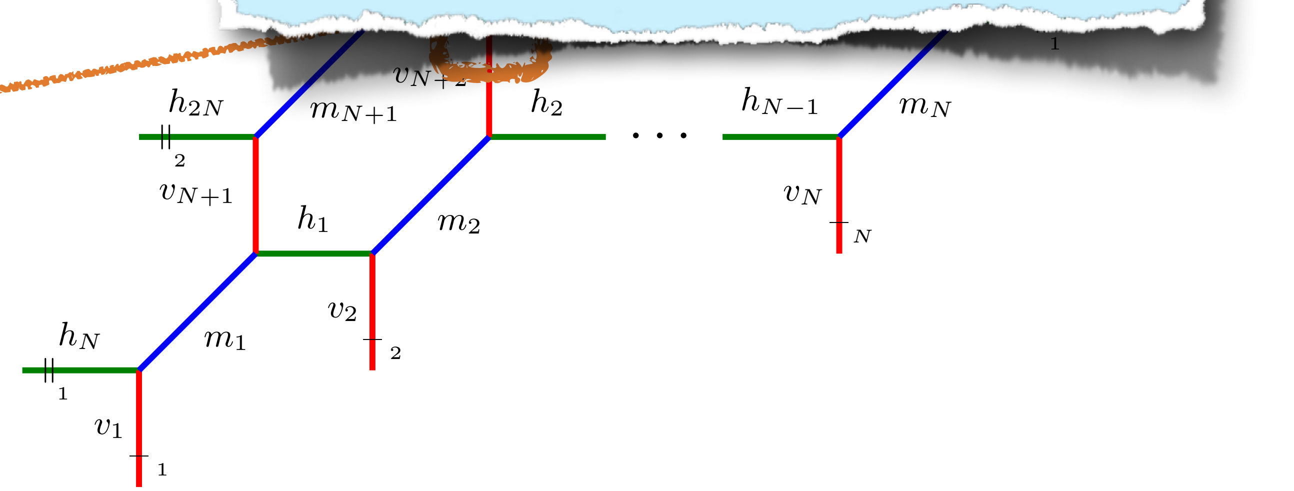
$$|\mu| = \sum_{i=1}^{\ell} \mu_i$$



$$||\mu||^2 = \sum_{i=1}^{\ell} \mu_i^2$$

$s_{\mu/\eta}$ skew Schur function





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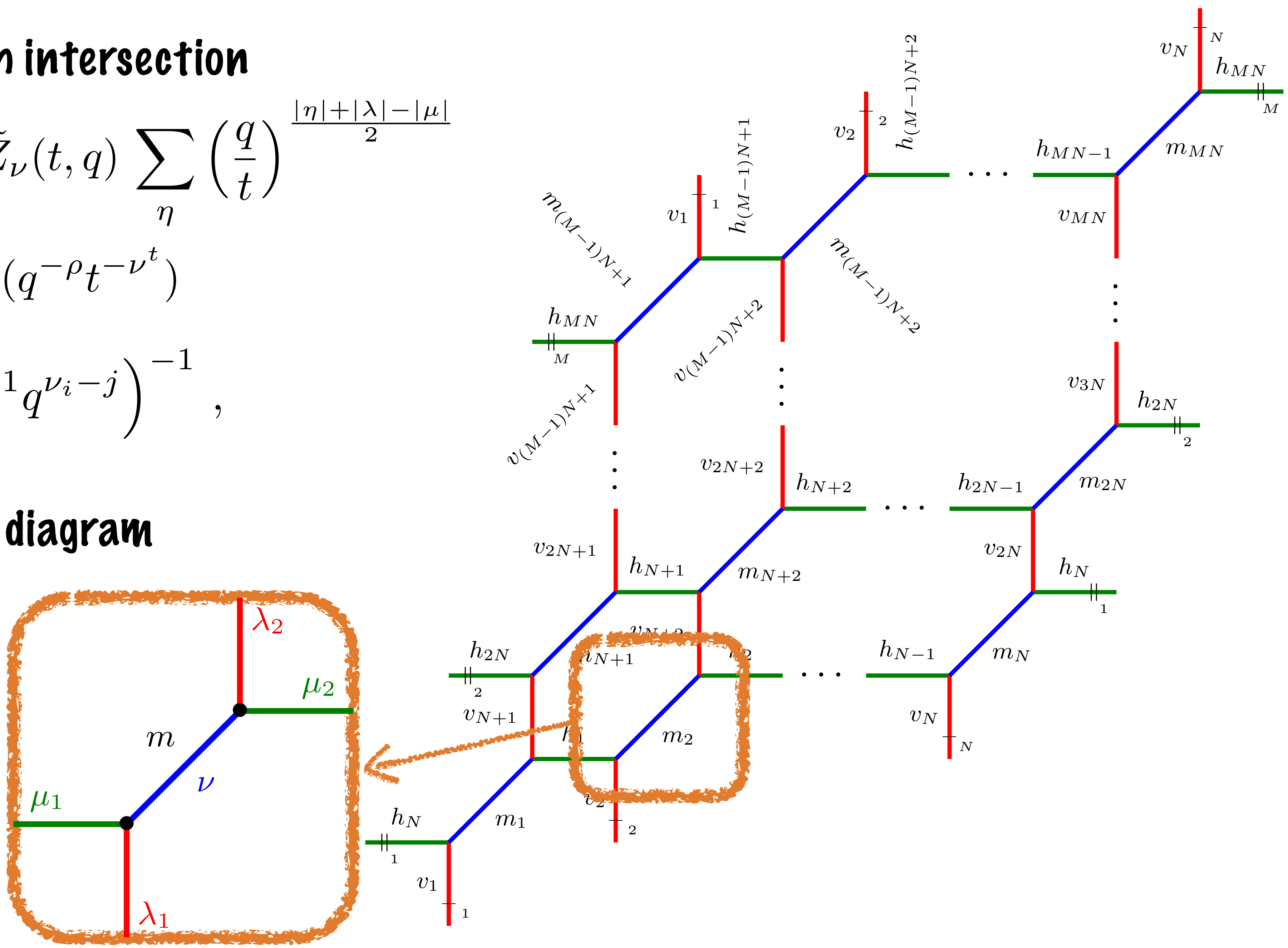
- glue vertices according to web diagram



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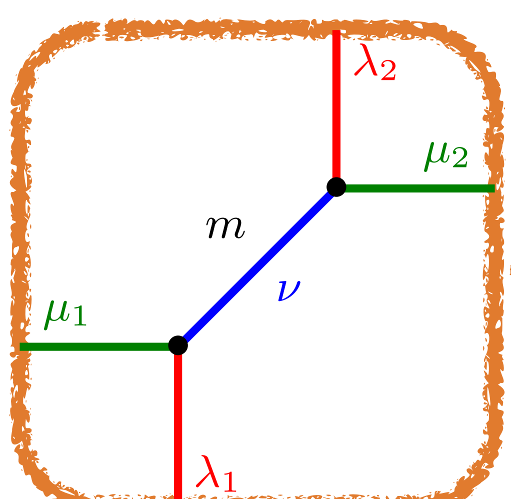

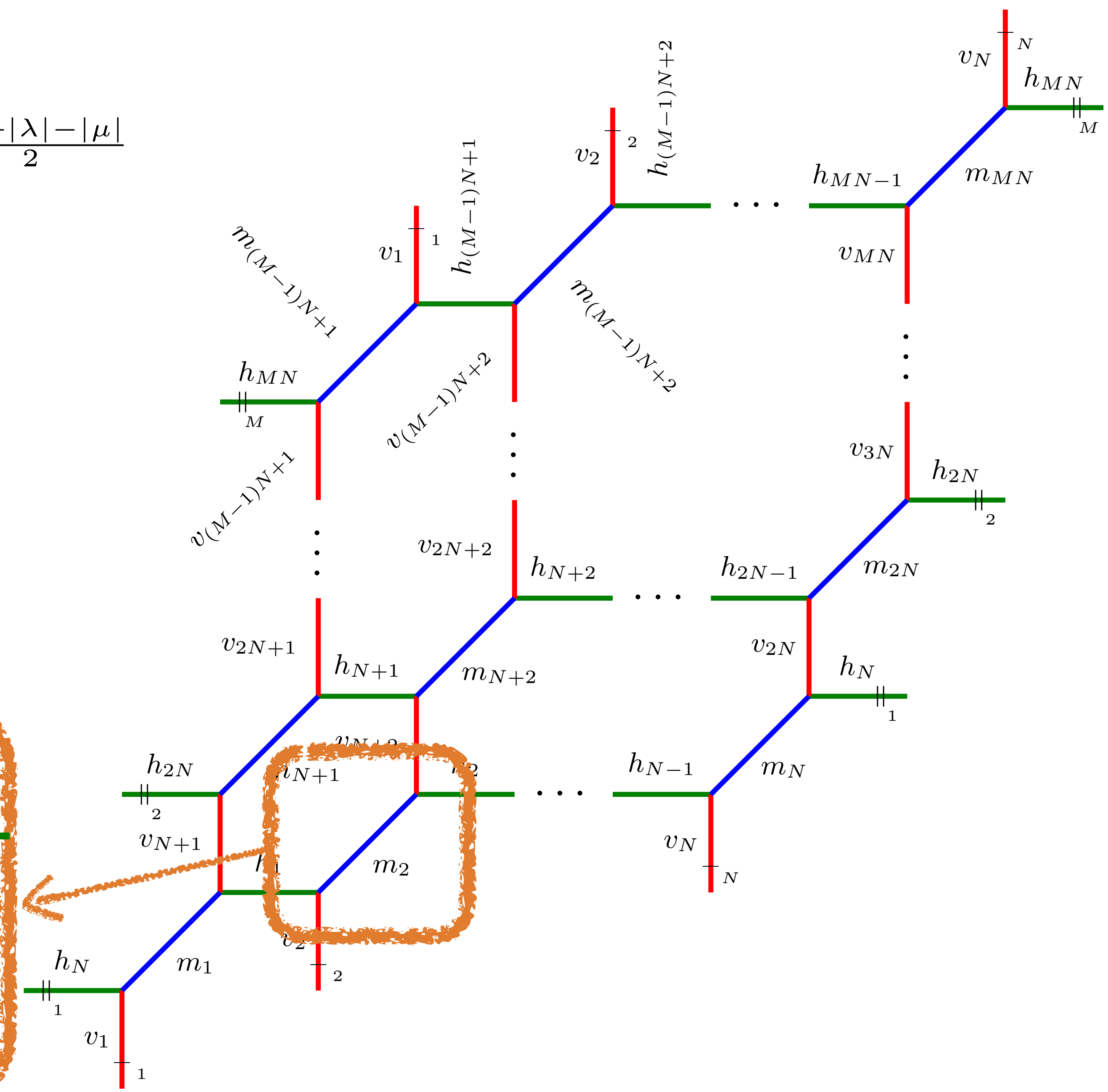

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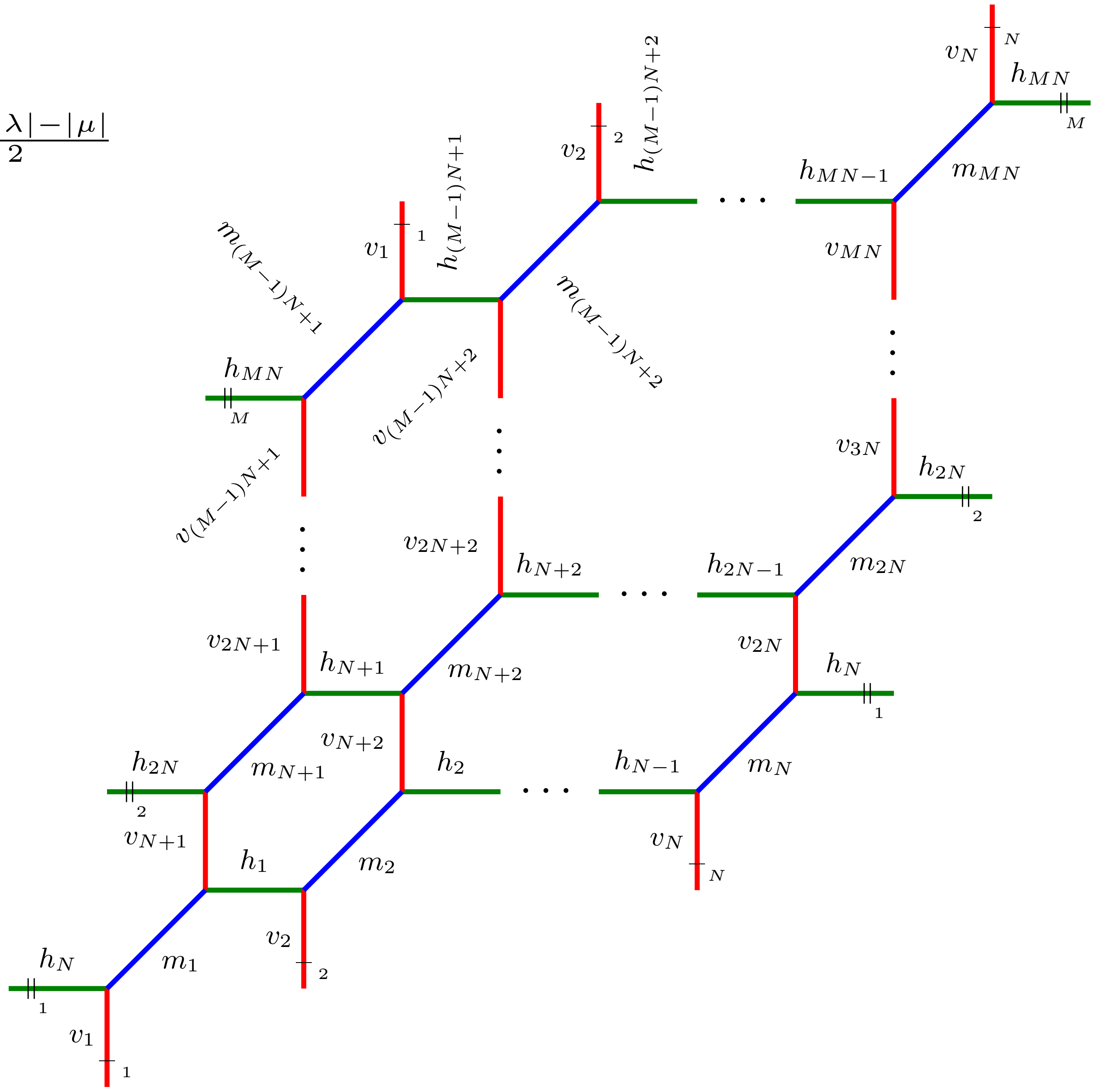
- choose preferred direction



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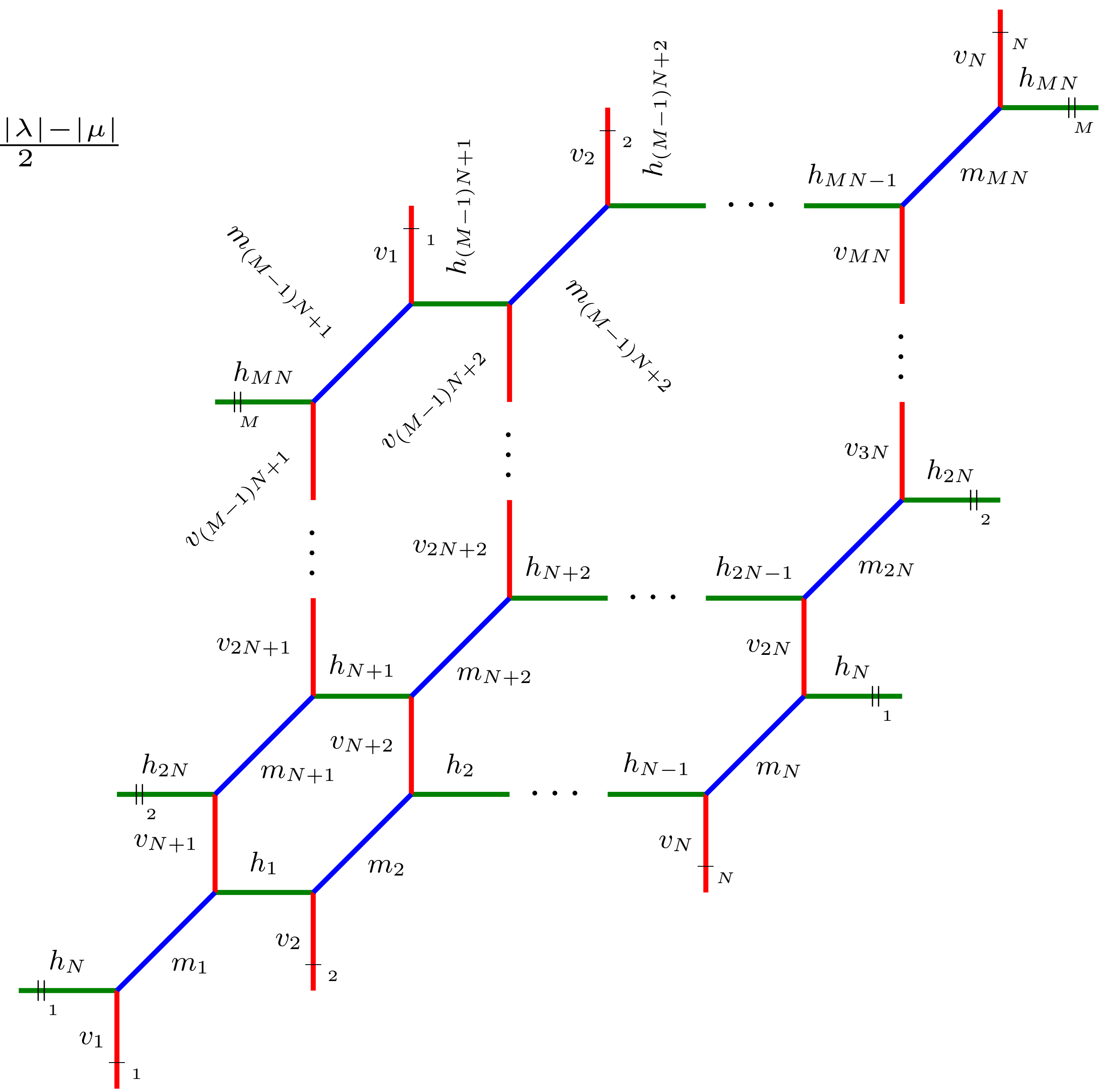
-) choose **preferred direction**

must be common to all vertices of diagram



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Different choices of preferred direction afford different (but equivalent) expansions:

$$\begin{aligned}
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Equalities of partition functions implies dualities among different gauge theories: Triality

[Bastian, SH, Iqbal, Rey 2017]

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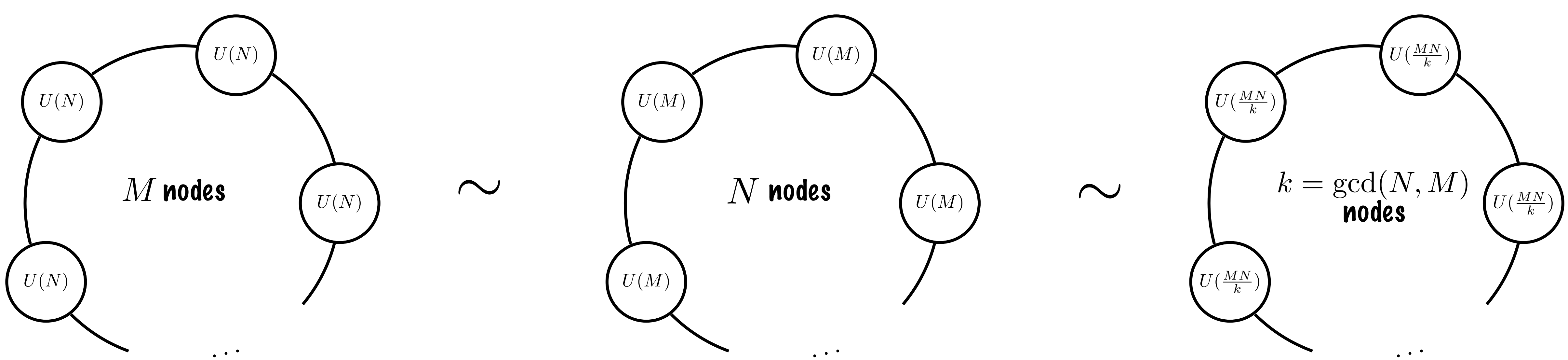

Further Dualities from $X_{N,M}$

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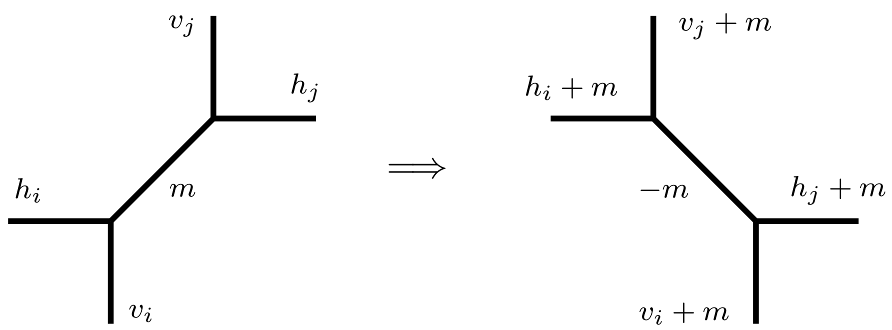

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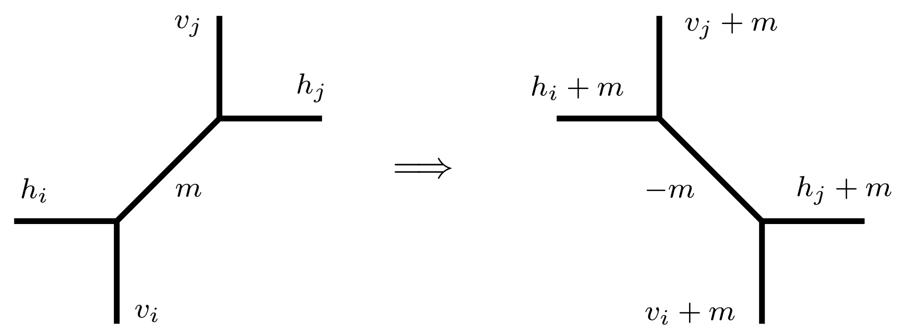

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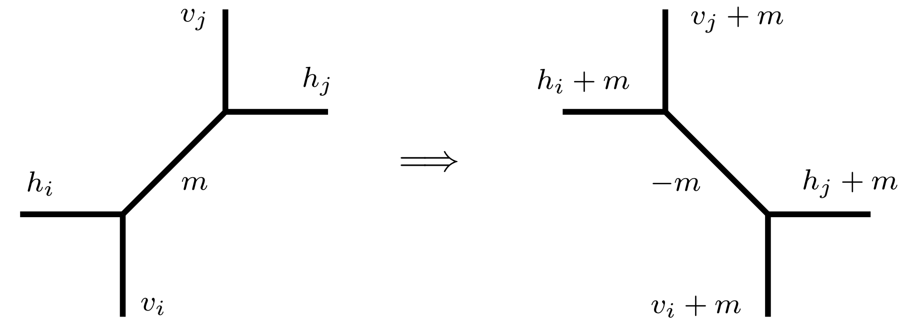

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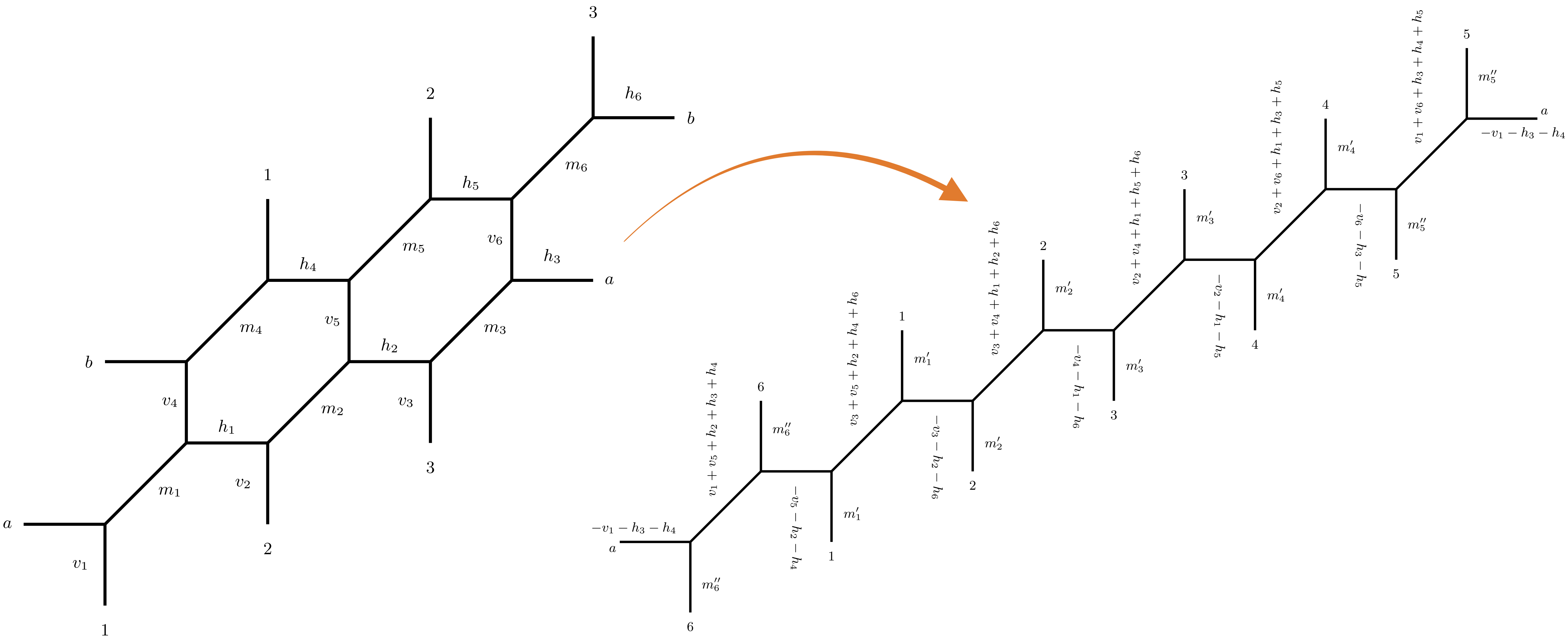

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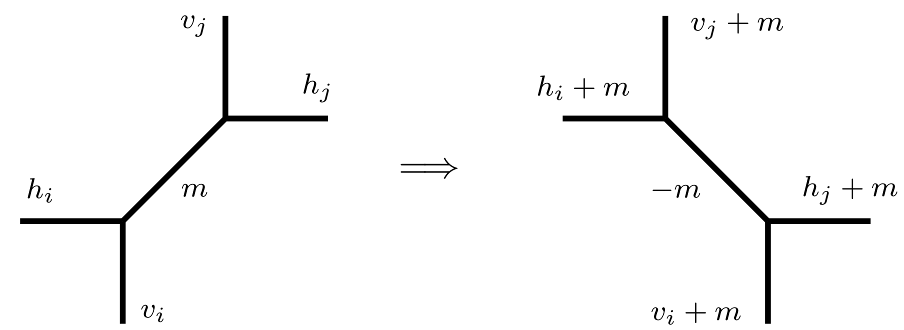

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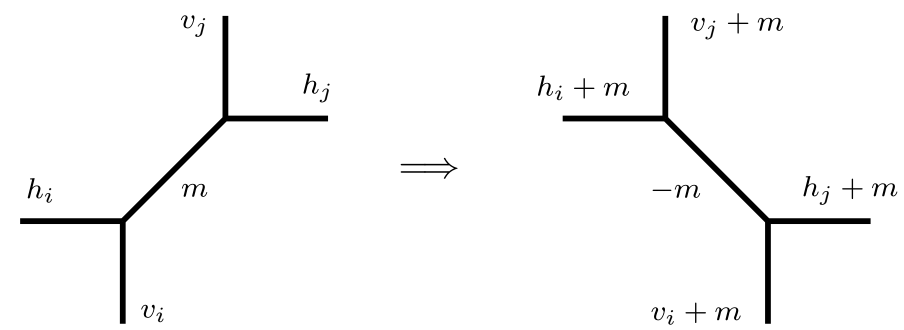

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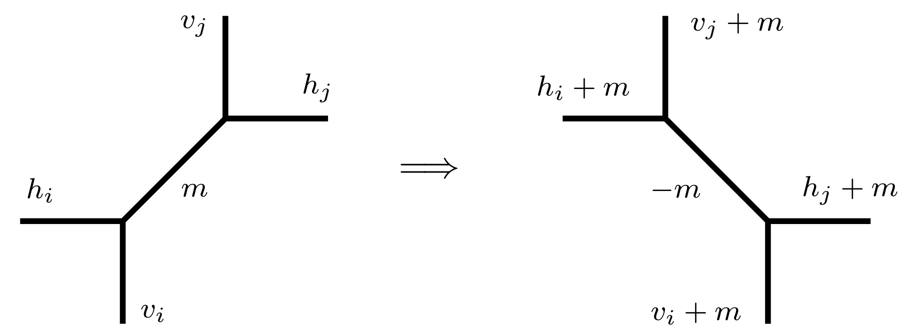

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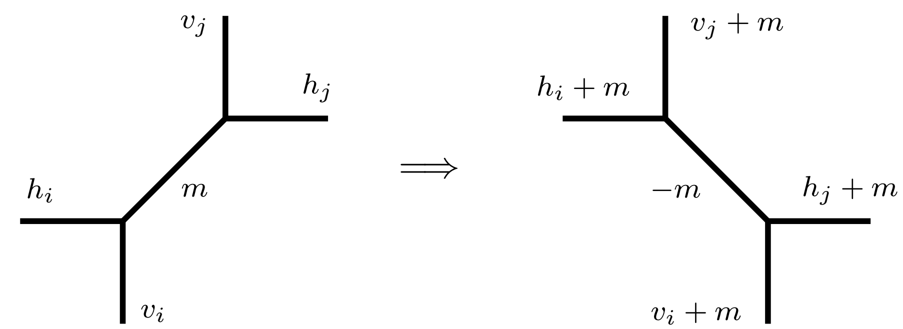

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Further dualities among larger classes of gauge symmetries

Network of Dual Theories

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Extended moduli space of $X_{N,M}$:

$$X_{N,M} \sim X_{N',M'} \qquad \textbf{for} \qquad \begin{aligned} NM &= N'M' \\ \gcd(N,M) &= \gcd(N',M') \end{aligned}$$

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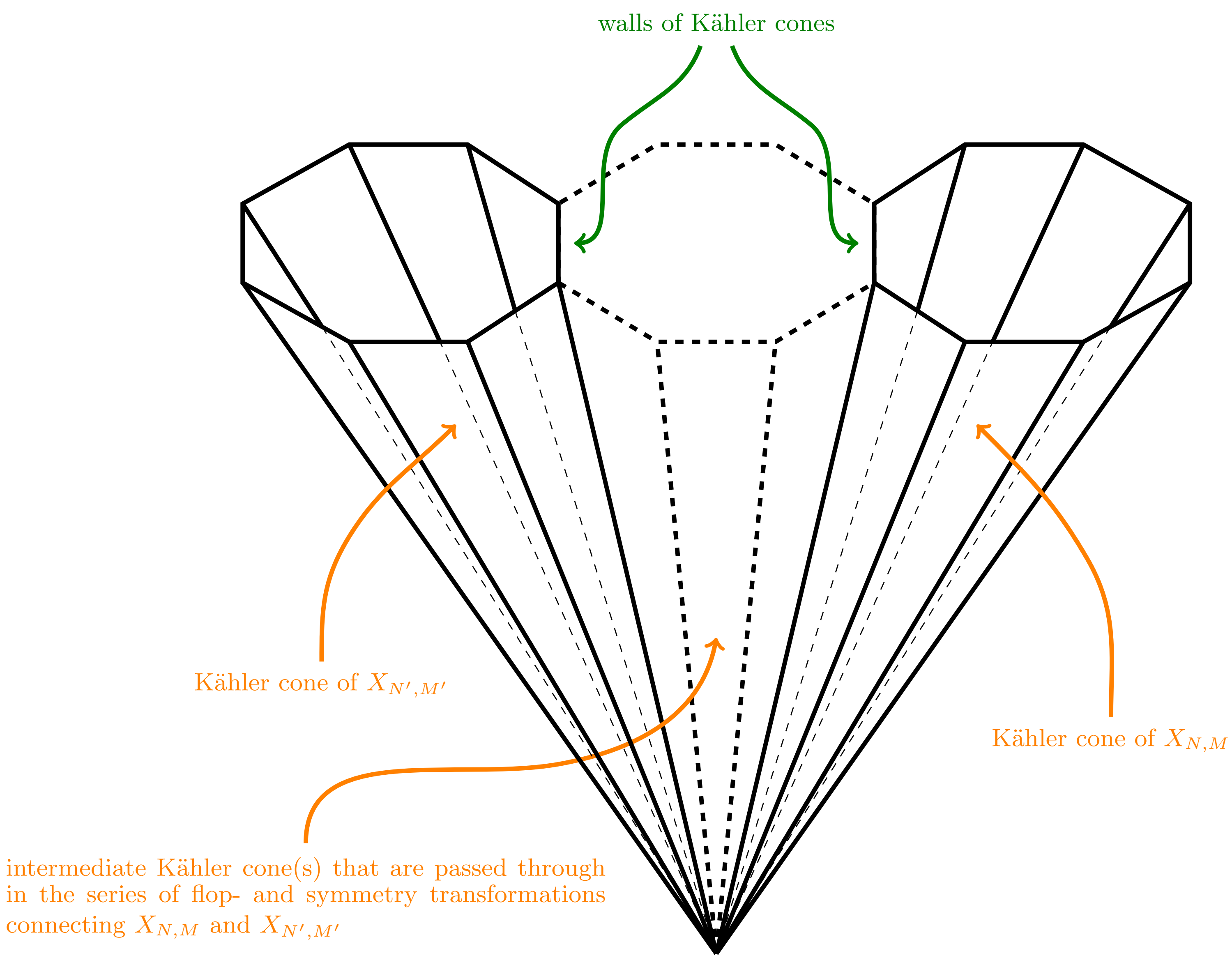

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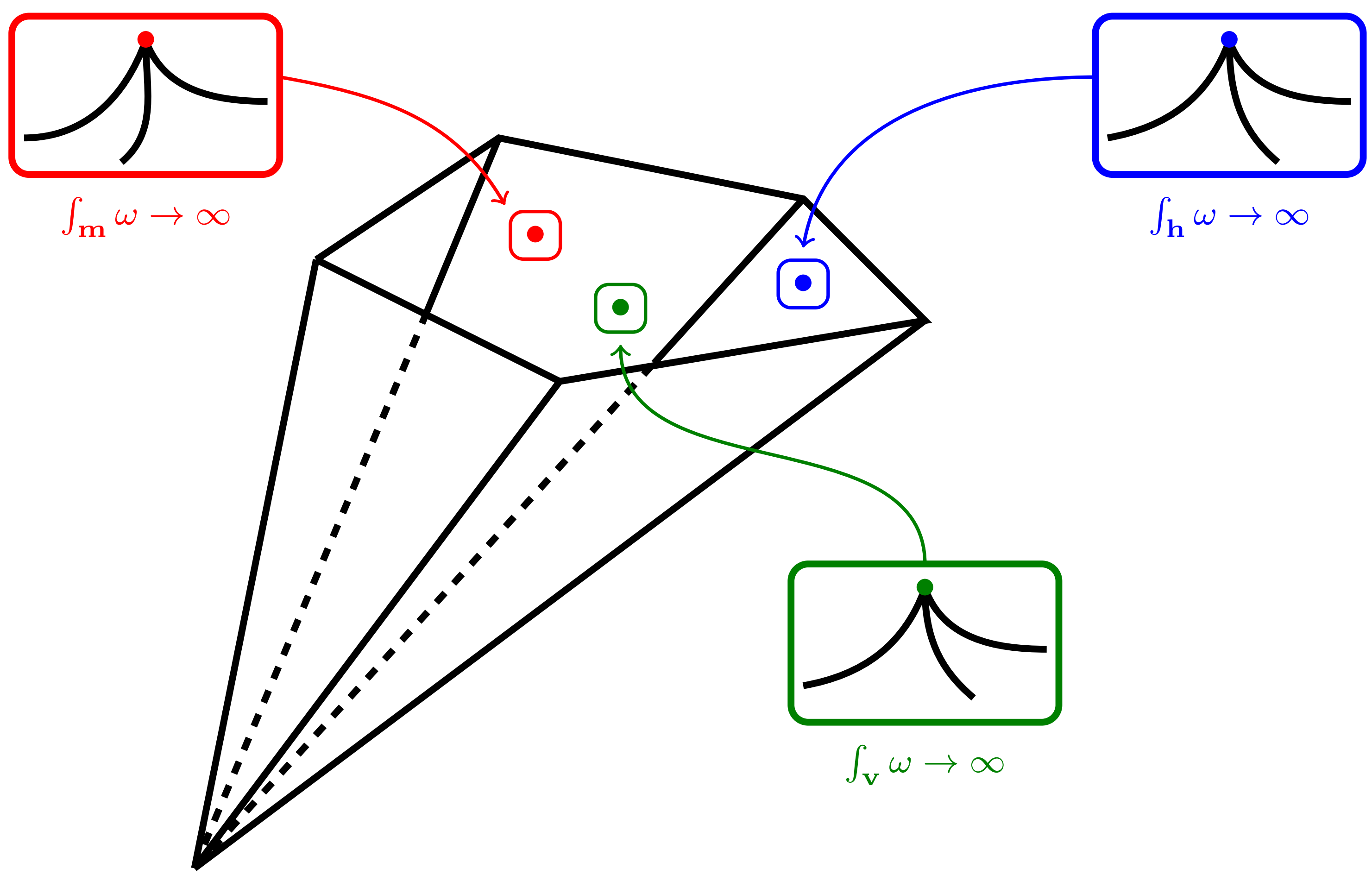

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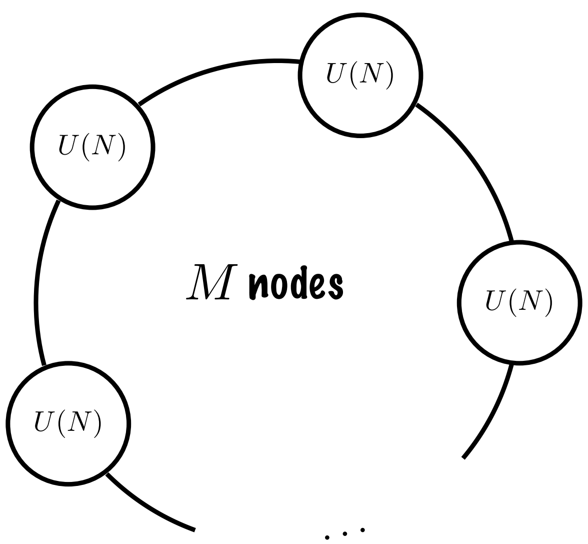
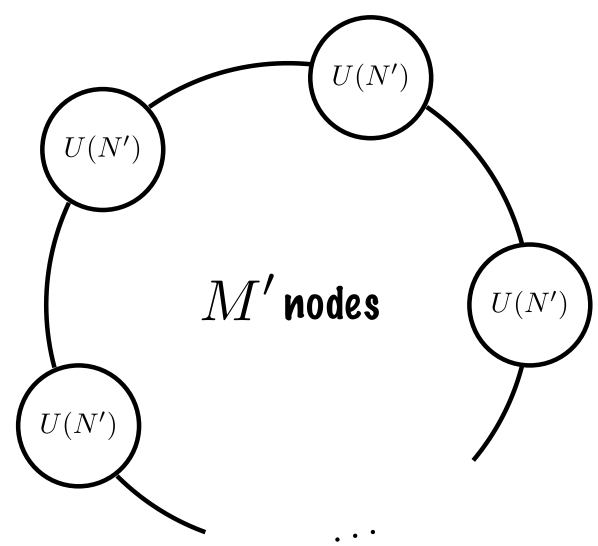


$\sim$



for any (N',M') with

$$NM = N'M'$$
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Dihedral Symmetries of Configurations (N,1)

Web of dualities among different theories can be turned into symmetries for individual theories

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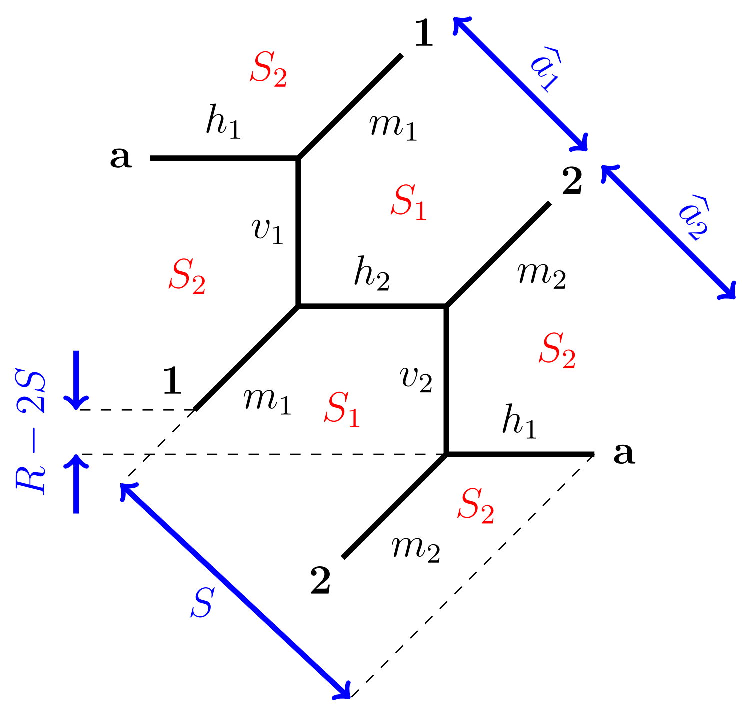

$$\widehat{a}_1 = v_1 + h_2\,, \qquad \widehat{a}_2 = v_2 + h_1\,,$$

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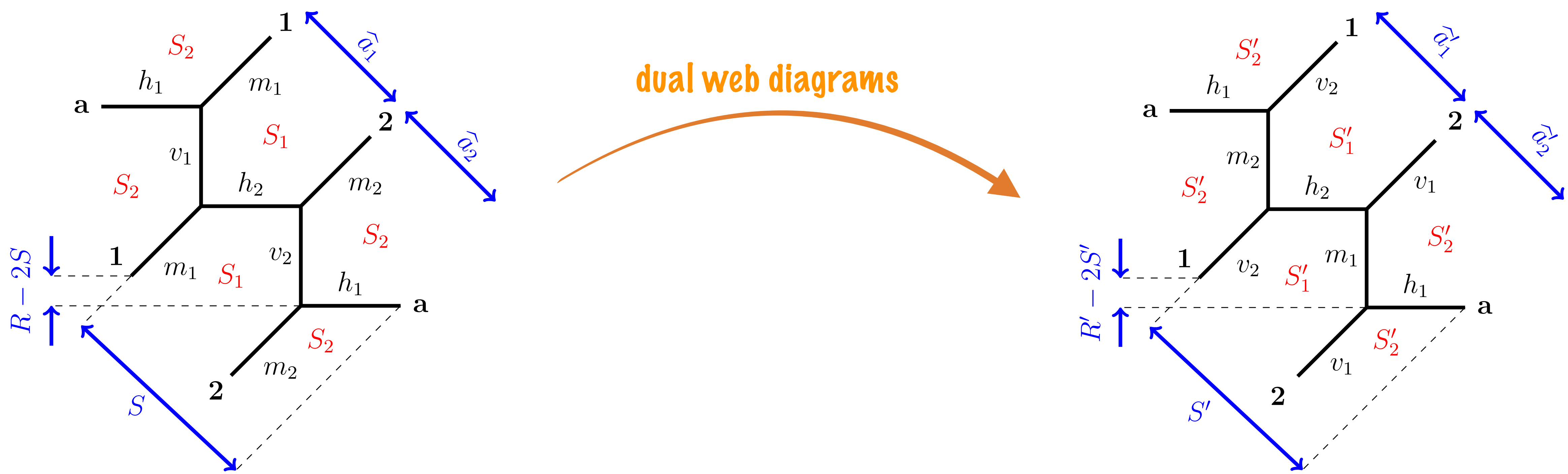

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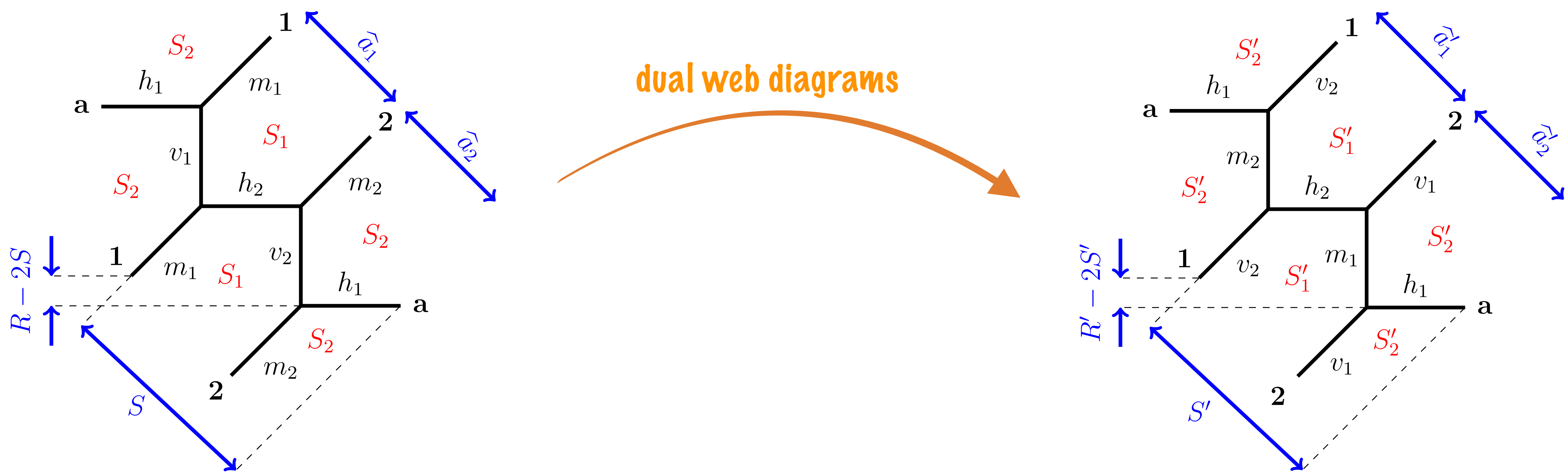

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Implies the following symmetry of the partition function:

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \\ S \\ R \end{pmatrix} = G_1 \cdot \begin{pmatrix} \widehat{a}'_1 \\ \widehat{a}'_2 \\ S' \\ R' \end{pmatrix}$$

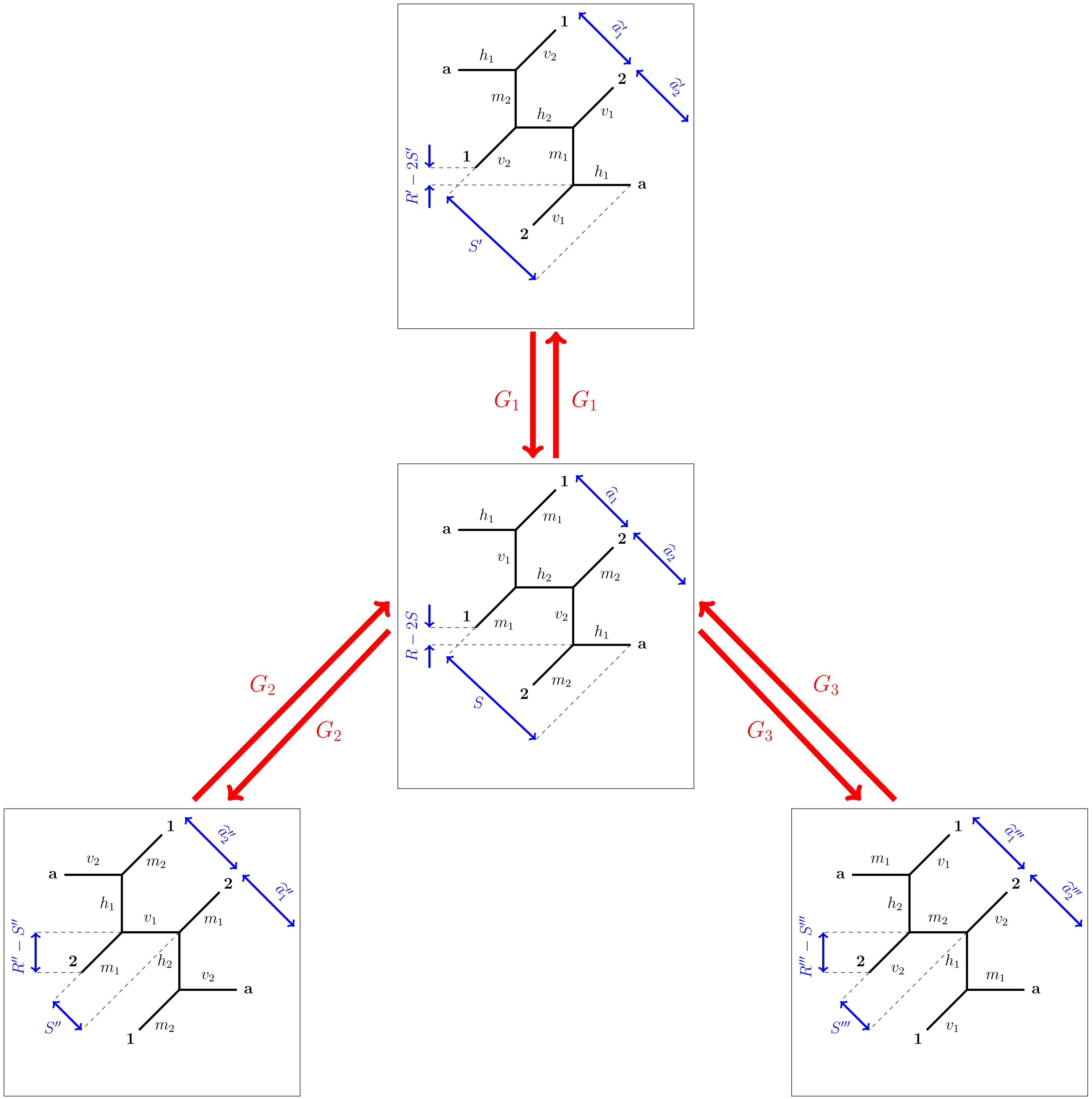
where

$$G_1 = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$\det G_1 = 1$$
$$G_1 \cdot G_1 = \mathbb{1}_{4\times 4}$$

Generalising to include other duality transformations:



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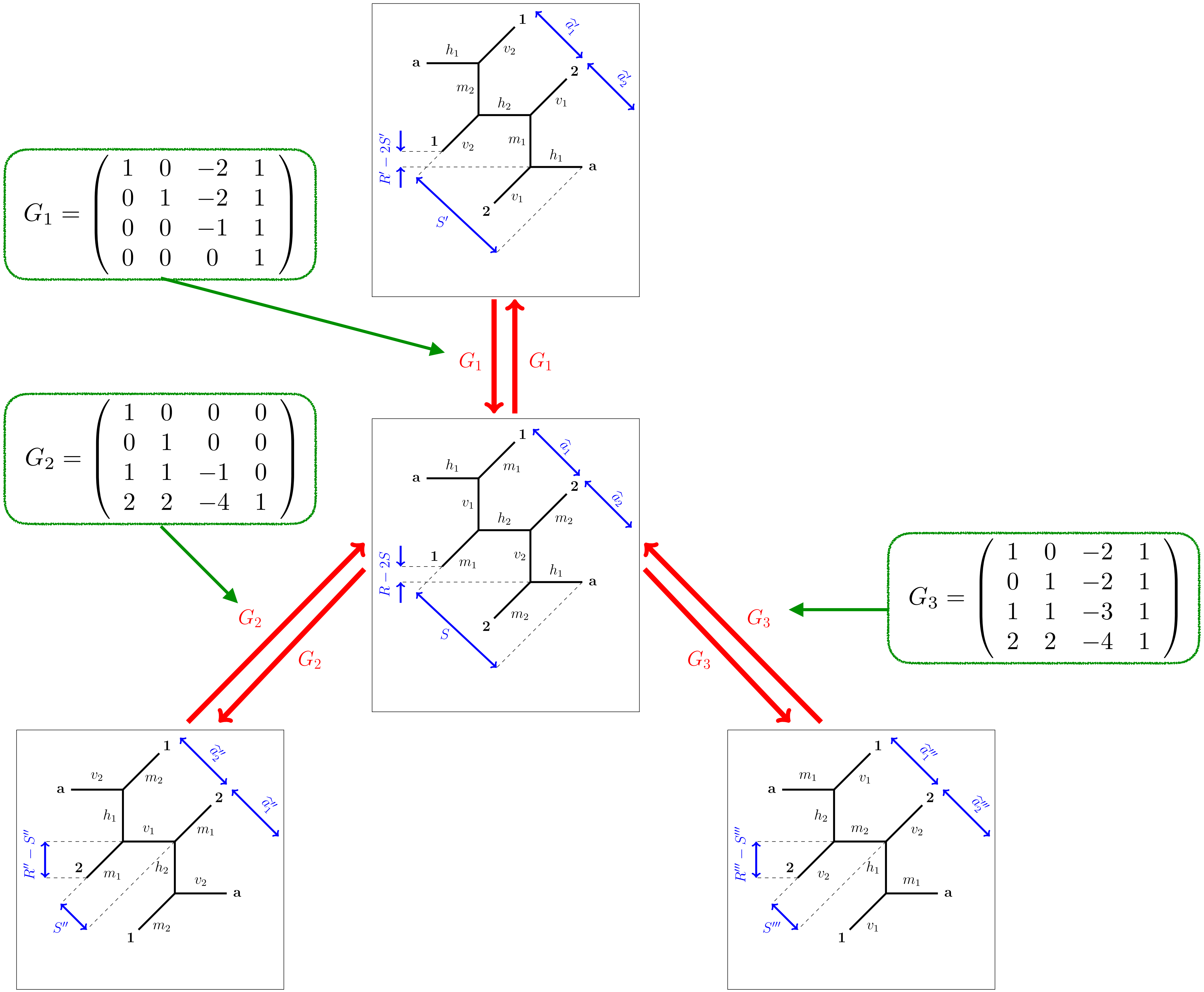

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G_2	G_2	G_3	$\mathbb{1}_{4\times4}$	G_1
G_3	G_3	G_2	G_1	$\mathbb{1}_{4\times4}$

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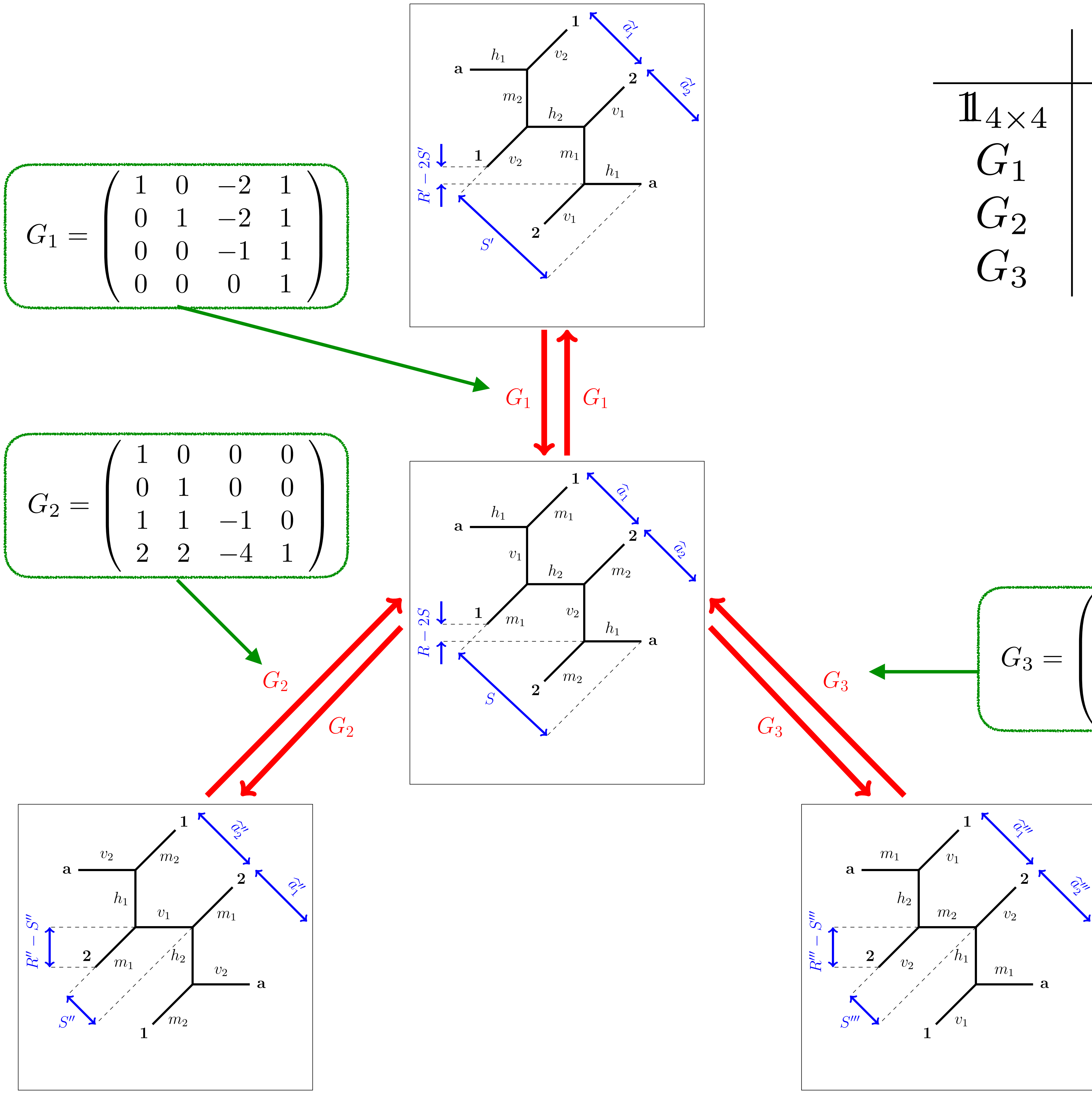

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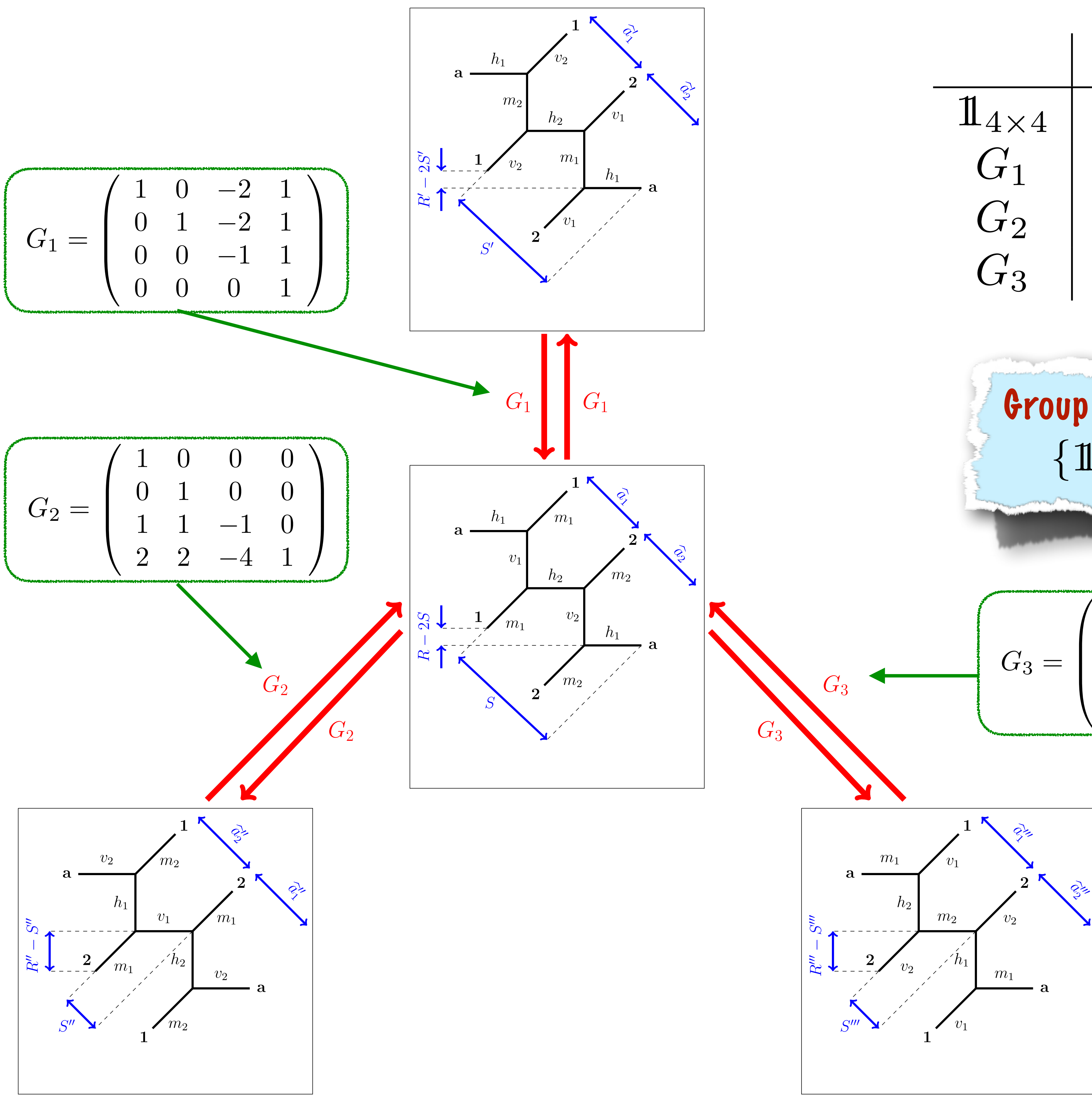

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G_3	G_3	G_2	G_1	$\mathbb{1}_{4\times4}$

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$$\{\mathbb{1}_{4\times4}, G_1, G_2, G_3\} \cong \mathrm{Dih}_2$$

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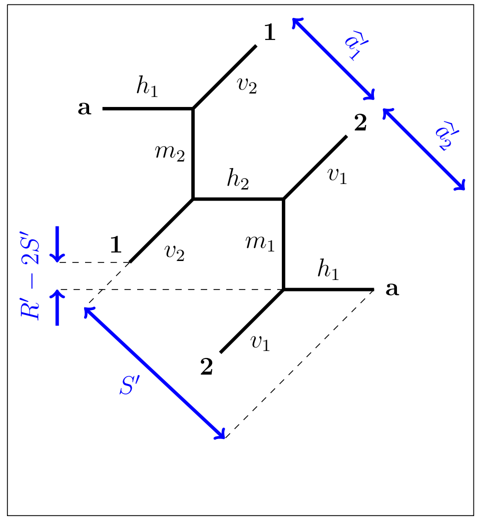

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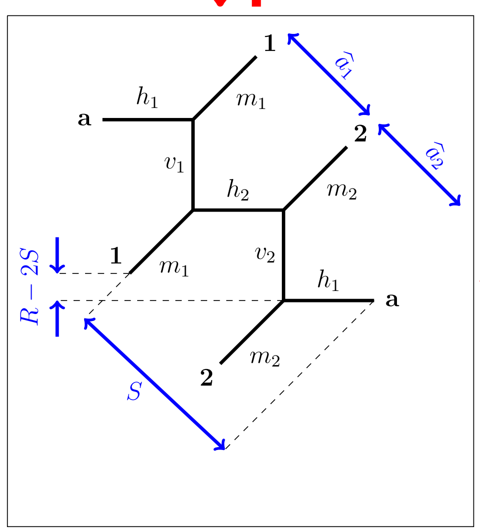

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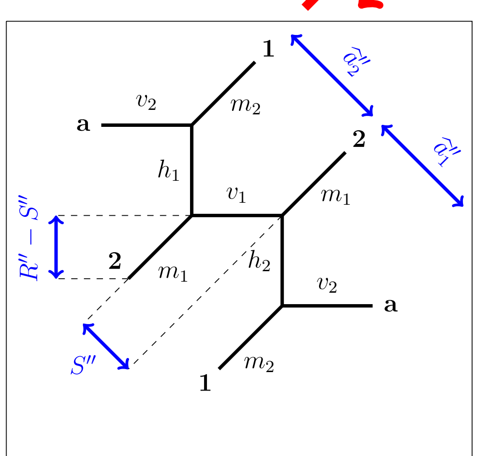

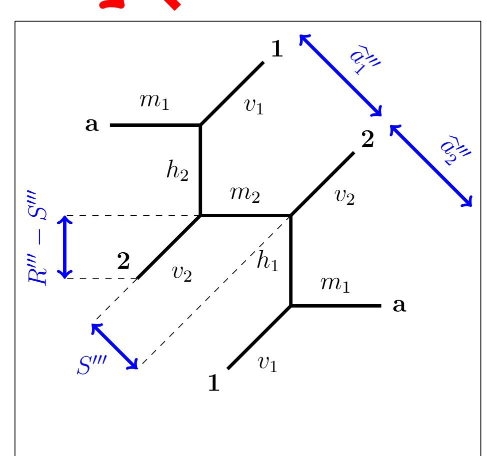

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Symmetries act non-perturbatively from the perspective of any of the gauge theories

Generalisation to (N,1): Symmetry group

$$\mathbb{G}(N) \times \mathrm{Dih}_N$$

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Explicitly

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with the $(N+2) \times (N+2)$ **matrices**

$$\mathcal{G}_2(N) = \begin{pmatrix} & & & 0 & 0 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ N & \cdots & N & -2N & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{G}_2'(N) = \begin{pmatrix} & & & -2 & 1 \\ & \mathbb{1}_{N\times N} & & \vdots & \vdots \\ & & & -2 & 1 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Action on the Free Energy: $(N,1)$

Fourier Expansion of the Free Energy:

$$F_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2}) = \ln\, \mathcal{Z}_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2}) = \sum_{s_1,s_2=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_N}^{\infty} \sum_{k\in\mathbb{Z}} \epsilon_1^{s_1-1} \epsilon_2^{s_2-1} f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,n}\, Q^{i_1}_{\widehat{a}_1} \ldots Q^{i_N}_{\widehat{a}_N}\, Q^k_S\, Q^n_R$$

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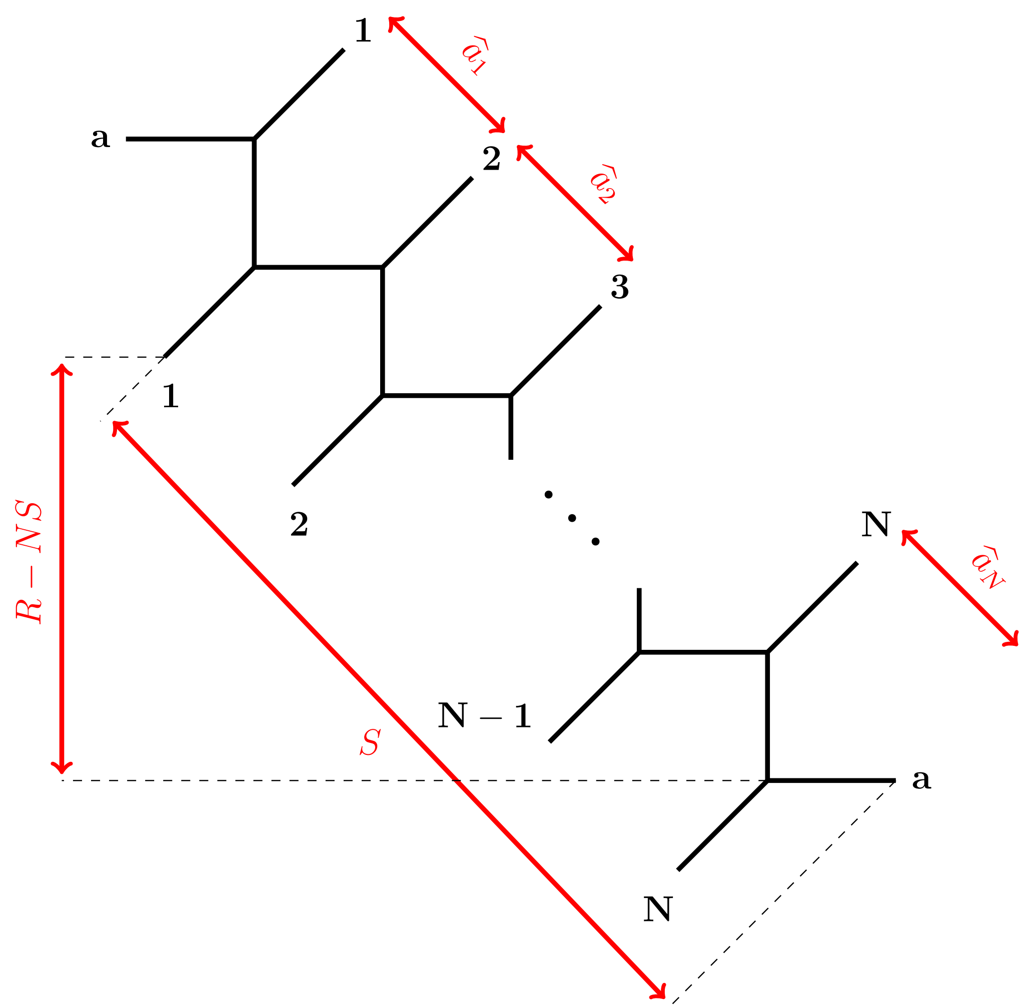

Notation:

$$Q_{\widehat{a}_i} = e^{2\pi i \widehat{a}_i} \qquad \rho = \sum_{i=1}^{N} \widehat{a}_i$$

$$Q_S = e^{2\pi i S}$$

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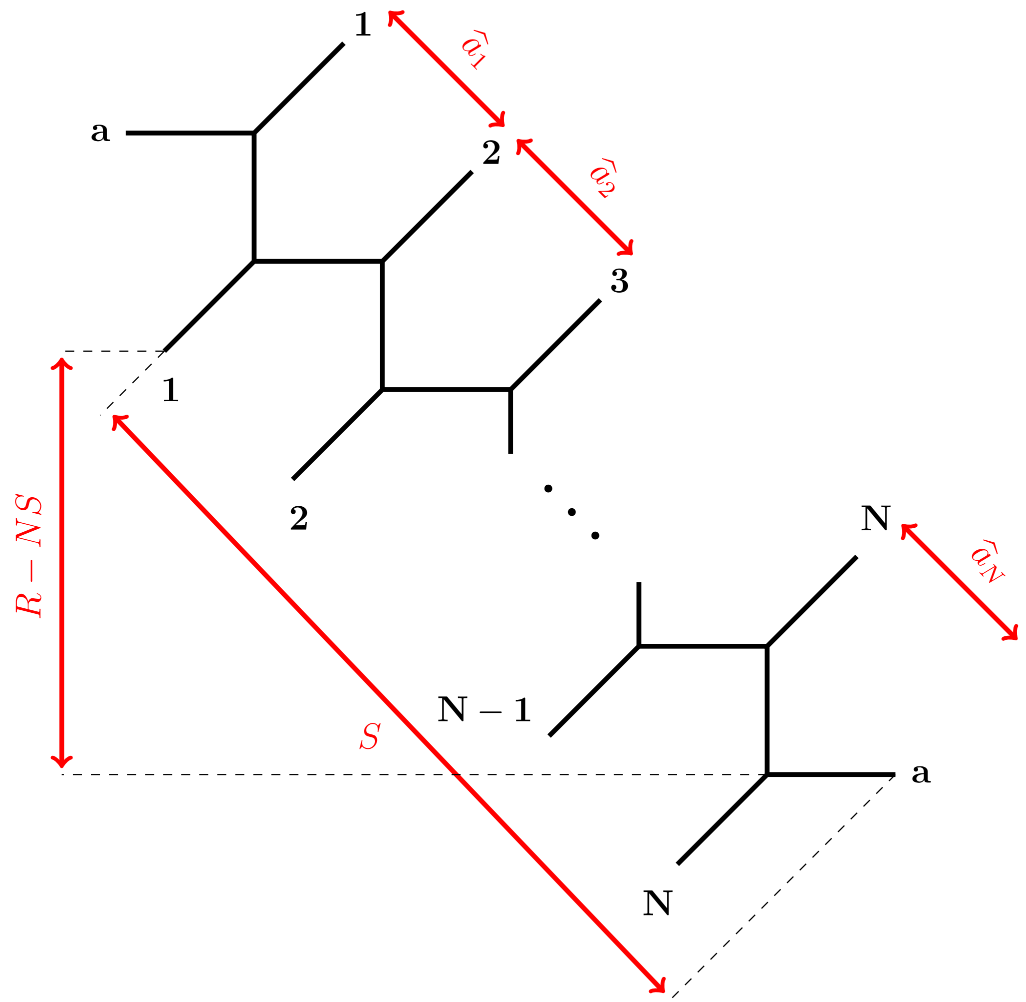

Action of $\mathbb{G}(N)\times\mathrm{Dih}_N$ on Fourier coefficients

$$f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,n} = f^{(s_1,s_2)}_{i'_1,\ldots,i'_N,k',n'}$$

for

$$(i'_1,\ldots,i'_N,k',n')^T = G^T\cdot(i_1,\ldots,i_N,k,n)^T$$

$$G\in\mathbb{G}(N)\times\mathrm{Dih}_N$$

Notation:

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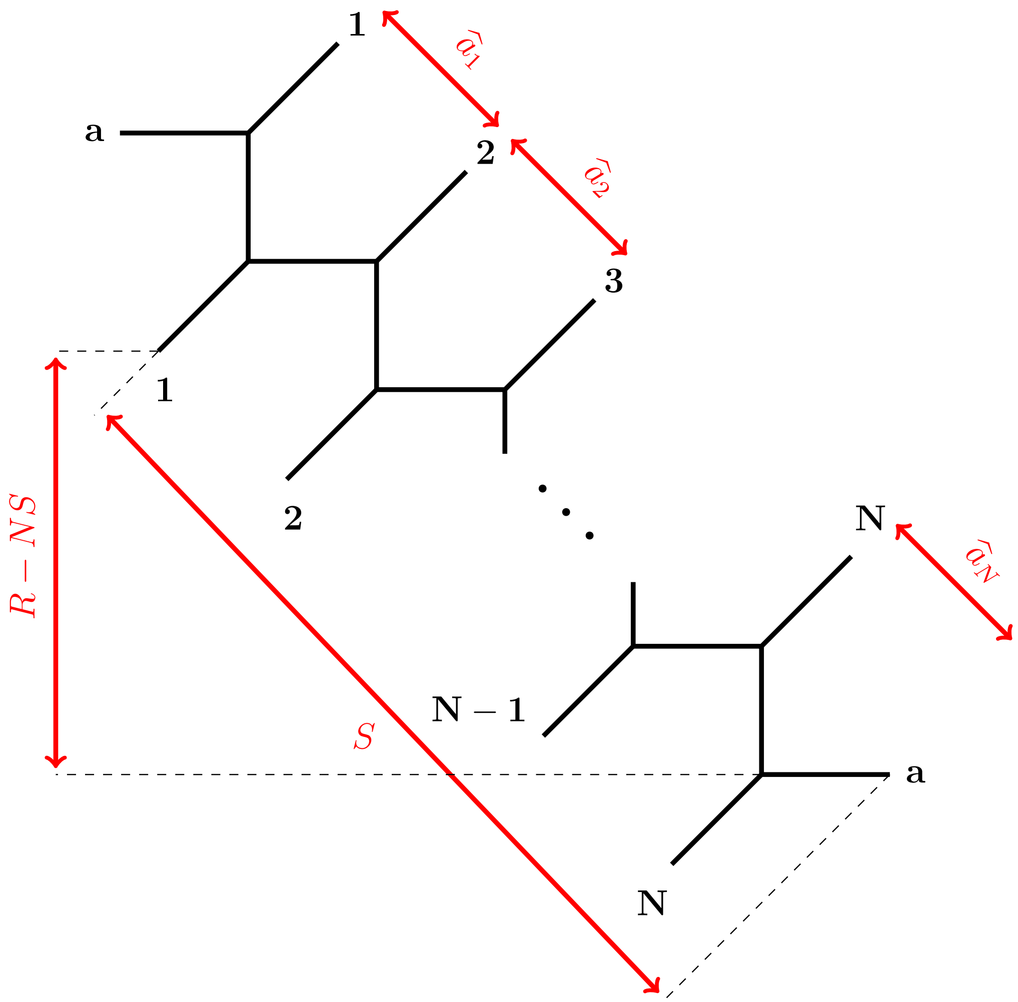
Action on the Free Energy: $(N, 1)$

Fourier Expansion of the Free Energy:

$$F_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2}) = \ln\, \mathcal{Z}_{N,1}(\widehat{a}_i, S, R; \epsilon_{1,2}) = \sum_{s_1,s_2=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_N}^{\infty} \sum_{k\in\mathbb{Z}} \epsilon_1^{s_1-1} \epsilon_2^{s_2-1} f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,n}\, Q^{i_1}_{\widehat{a}_1} \ldots Q^{i_N}_{\widehat{a}_N}\, Q^k_S\, Q^n_R$$



Action of $\mathbb{G}(N) \times \mathrm{Dih}_N$ on Fourier coefficients

$$f^{(s_1,s_2)}_{i_1,\ldots,i_N,k,n} = f^{(s_1,s_2)}_{i'_1,\ldots,i'_N,k',n'}$$

for

$$(i'_1,\ldots,i'_N,k',n')^T = G^T \cdot (i_1,\ldots,i_N,k,n)^T$$

$$G \in \mathbb{G}(N) \times \mathrm{Dih}_N$$

checked explicitly in numerous examples [SH, Bastian 2018]

Notation:

$$Q_{\widehat{a}_i} = e^{2\pi i \widehat{a}_i} \qquad \rho = \sum_{i=1}^{N} \widehat{a}_i$$

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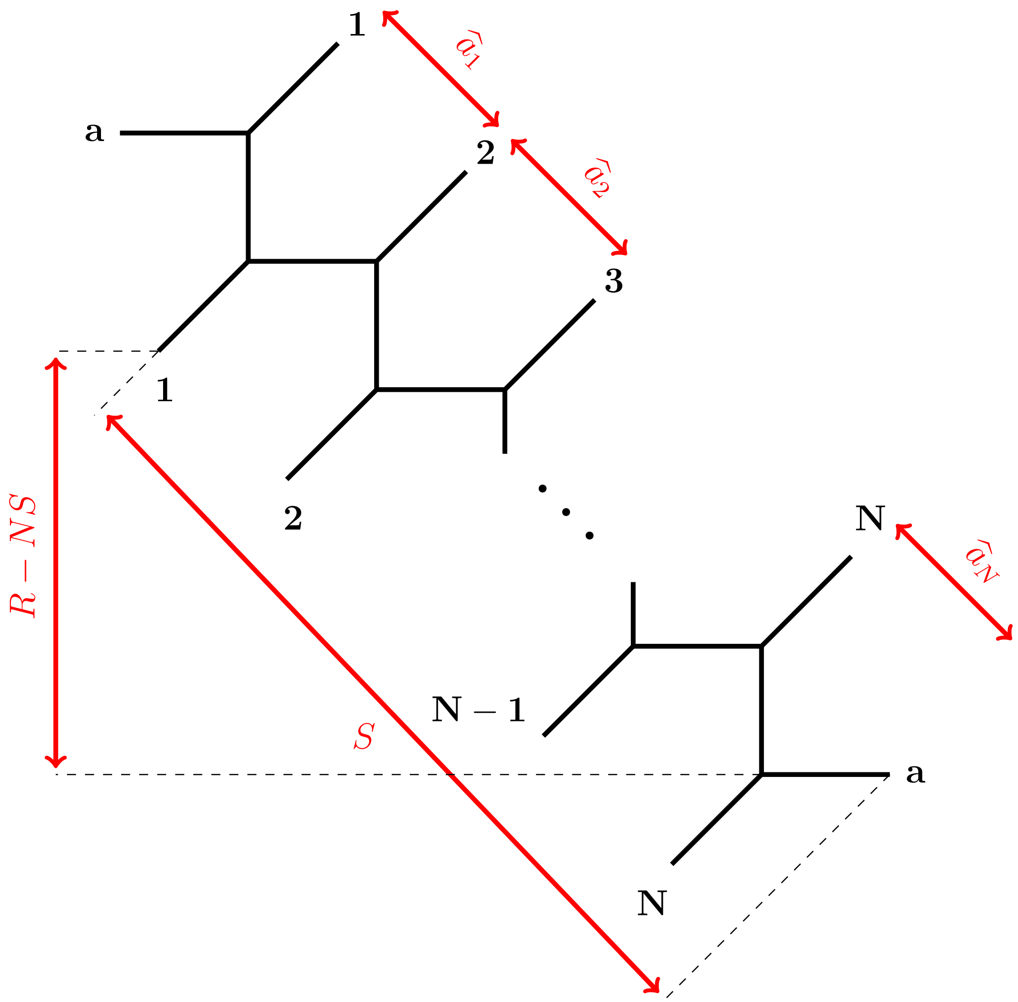

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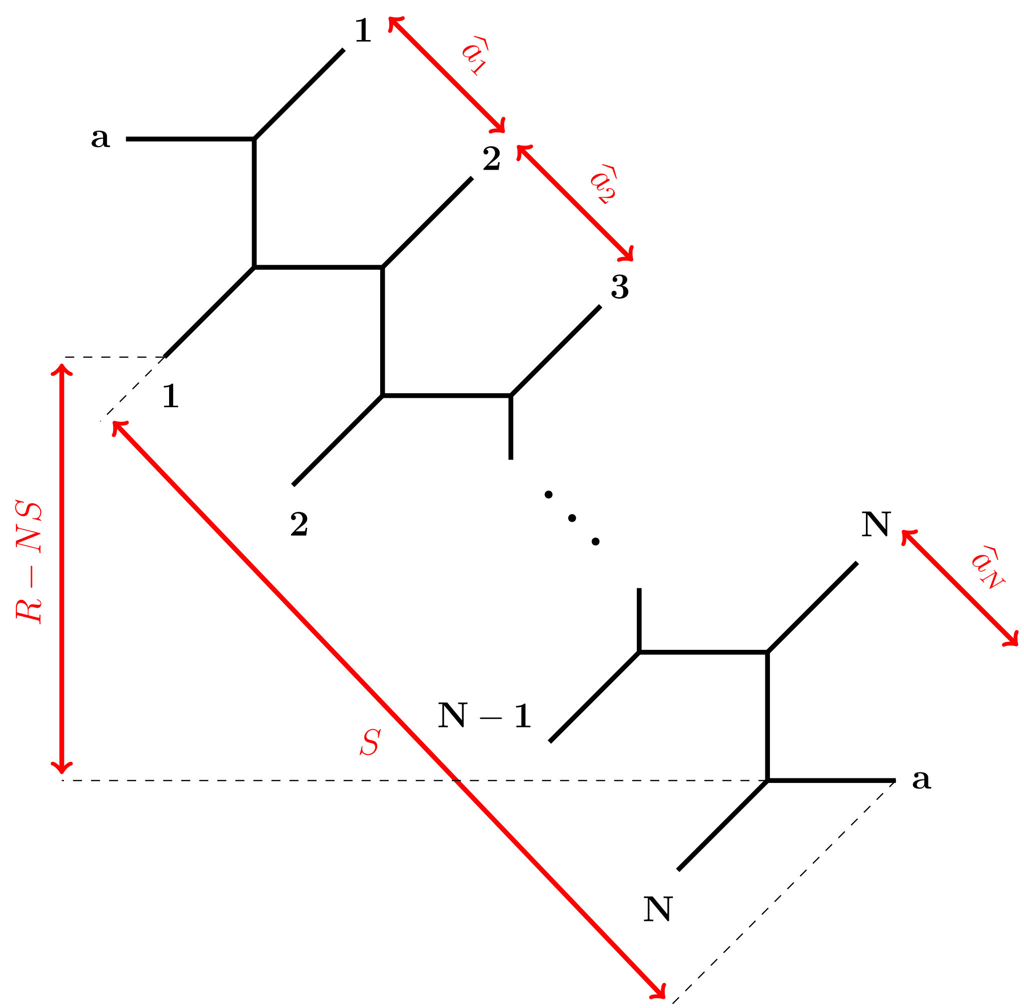

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Transformations also involve the instanton parameter — non-perturbative symmetries

Embedding into Paramodular Group

Combine parameters into period matrix of 2-torus

$$\Omega = \begin{pmatrix} R & S \\ S & \rho/N \end{pmatrix}$$

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$$\Sigma_N = \left\{ \begin{pmatrix} * & *N & * & * \\ * & * & * & */N \\ * & *N & * & * \\ *N & *N & *N & * \end{pmatrix} \in Sp(4,\mathbb{Q}) \middle| * \in \mathbb{Z} \right\}$$

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Leaves the reduced free energy invariant

[Bastian, SH 2019]

$$\mathcal{F}_{N,1}(R,S,\rho,\epsilon_{1,2}) = \sum_{k,n\geq 0} Q_\rho^k Q_R^n \prod_{i=1}^N \oint \frac{dQ_{\widehat{a}_i}}{Q_{\widehat{a}_i}^{k+1}} \ln \mathcal{Z}_{N,1}(\omega,\epsilon_{1,2})$$

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In the Nekrasov-Shatashvili limit (for $\epsilon_2 \to 0$), enhancement of Σ_N

$\Sigma_N^* = \Sigma_N \cup \Sigma_N V_N \subset Sp(4,\mathbb{R})$ with $V_N = \begin{pmatrix} U_N & 0 \\ 0 & U_N^T \end{pmatrix}$ with $U_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N \\ 1 & 0 \end{pmatrix}$

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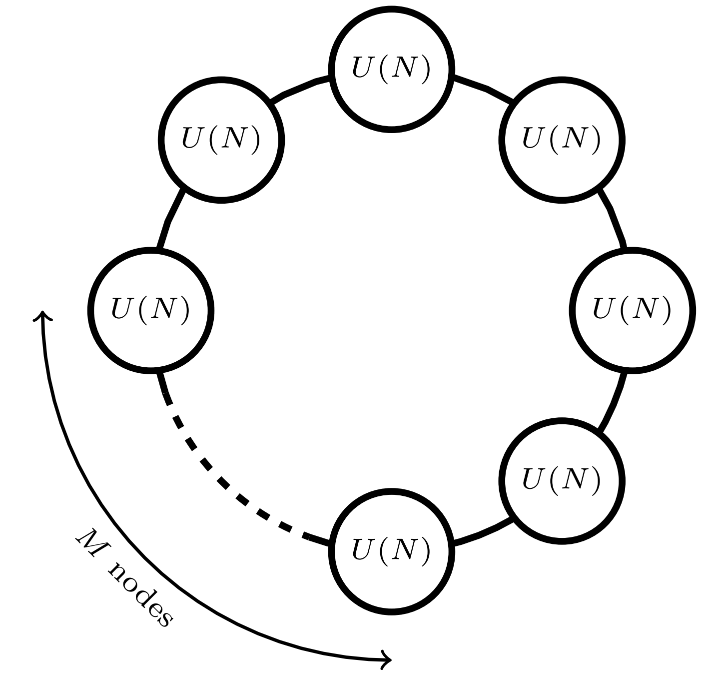
Corresponds to the symmetry $R \longleftrightarrow \rho$ of $\mathcal{F}_{N,1}$ [Bastian, SH 2019]

Generalisation to Configurations (N,M)

Start from the instanton expansion of $[U(N)]^M$ gauge theory:



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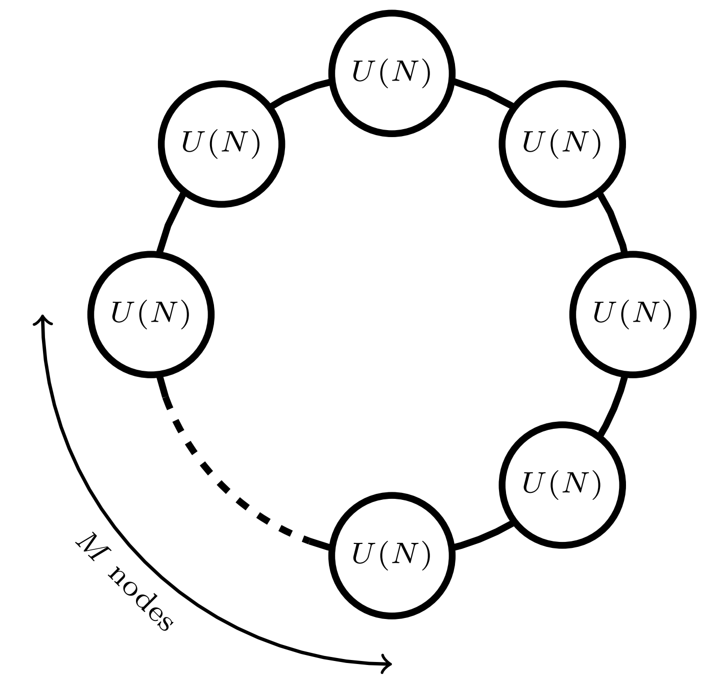

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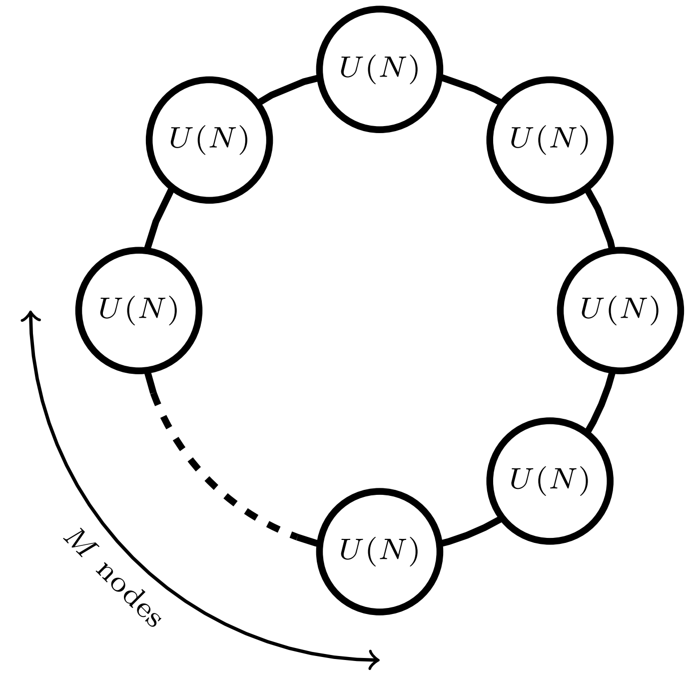

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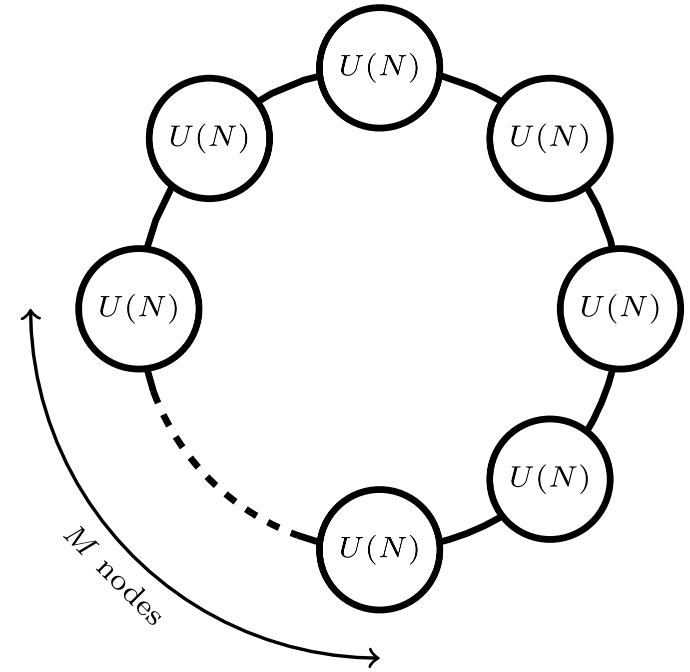

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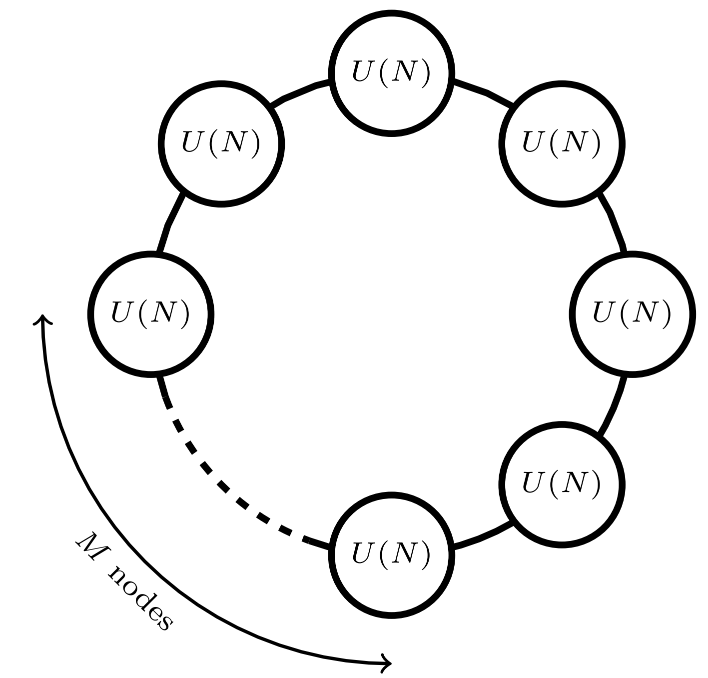

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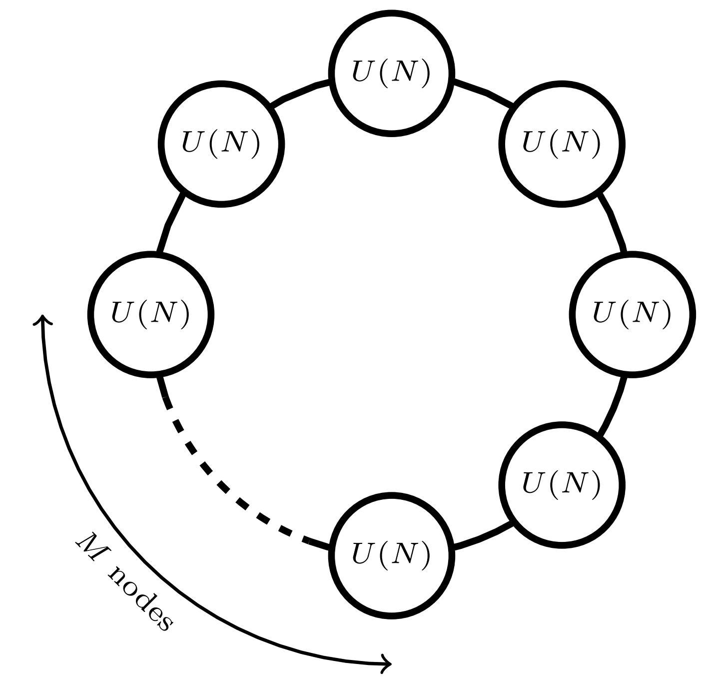

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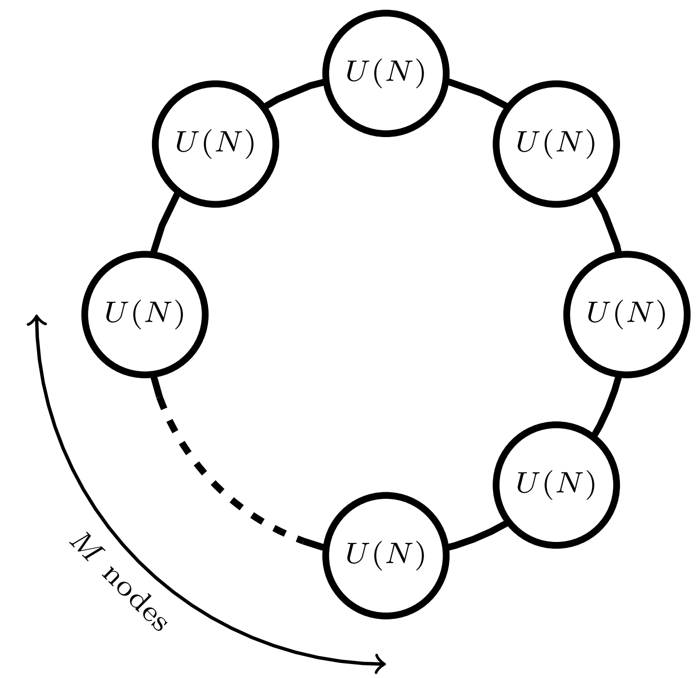

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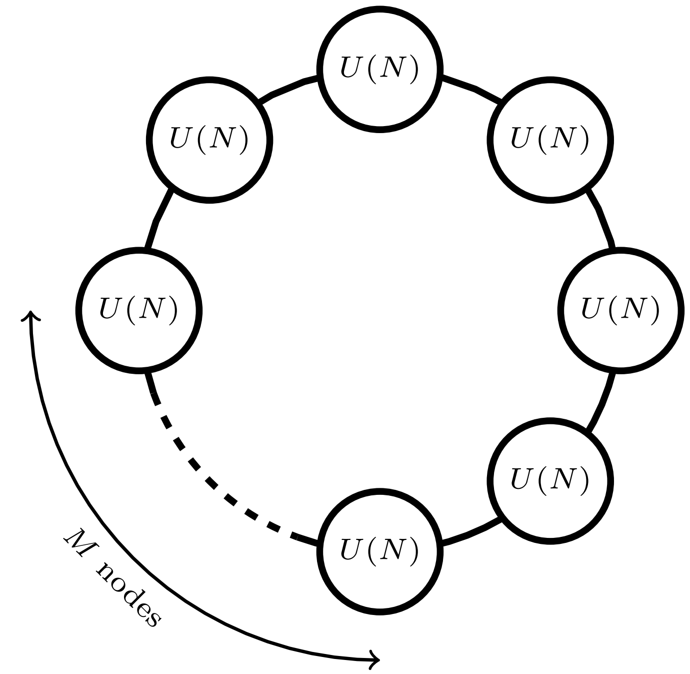

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Found 6 different types of symmetries that can be arranged into 6 involutions $\mathfrak{t}_{i=1,\ldots,6}$ with

$$\mathfrak{t}_i \circ \mathfrak{t}_i = \mathbb{1} \qquad \text{and} \qquad (\mathfrak{t}_i \circ \mathfrak{t}_j)^{\circ m_{ij}} = \mathbb{1}$$

[Filoche, SH, Kimura 2023]

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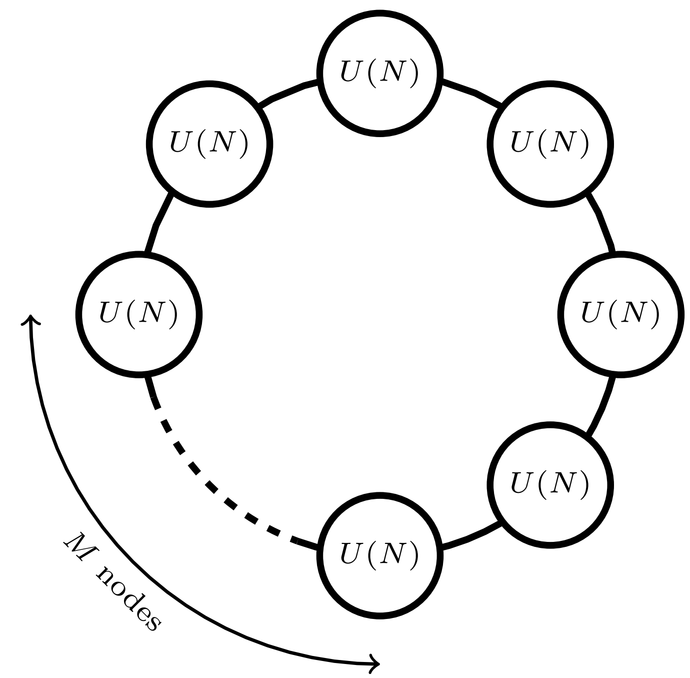

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Mathematica package **NPLSTsym.m** for explicit form of $\mathfrak{t}_{i=1,\ldots,6}$ [Filoche, SH, Kimura 2023]

$$(\mathfrak{t}_i \circ \mathfrak{t}_j)^{\circ m_{ij}} = \mathbb{1}$$

Braid-relations $(\mathfrak{t}_i \circ \mathfrak{t}_j)^{\circ m_{ij}} = \mathbb{1}$ **with Coxeter matrix** m_{ij}

$$m_{ij} = \begin{pmatrix} m^{ss} & m^{sd} \\ (m^{sd})^T & m^{dd} \end{pmatrix}_{ij}$$

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Dihedral group $\mathrm{Dih}_{m_{\mathfrak{a}\mathfrak{b}}}$

$m_{\mathfrak{a}\mathfrak{b}}$	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$M=1$	3	2	3	∞	∞	∞	∞	∞
$M=2$	2	3	$\underline{\infty}$	4	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$
$M=3$	3	$\underline{\infty}$	6	$\underline{\infty}$	$\underline{\infty}$	∞	$\underline{\infty}$	$\underline{\infty}$
$M=4$	∞	4	$\underline{\infty}$	6	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	∞
$M=5$	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	6	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$
$M=6$	∞	∞	∞	$\underline{\infty}$	$\underline{\infty}$	6	$\underline{\infty}$	$\underline{\infty}$
$M=7$	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	6	$\underline{\infty}$
$M=8$	∞	∞	$\underline{\infty}$	∞	$\underline{\infty}$	$\underline{\infty}$	$\underline{\infty}$	6

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Embedding of the t_i **into generalised para modular group under study**

[Filoche, SH, Kimura in preparation]

Physical interpretation at particular point in the moduli space

$$\vec{v} = (\vec{a}^{(1)}, \ldots, \vec{a}^{(M)}, \tau_1, \ldots, \tau_{M-1}, \tau, S, \rho)^T \longrightarrow \Big(\underbrace{\frac{\rho}{N}, \ldots, \frac{\rho}{N}}_{M(N-1)\text{-times}}, \underbrace{\frac{\tau}{M}, \ldots, \frac{\tau}{M}}_{M\text{-times}}, \tau, S, \rho \Big)^T$$

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Remaining parameters can be arranged in the period matrix of a genus 2 surface

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due to double elliptic vibration structure of $X_{N,M}$

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$t_{i=1,\dots,6}$ **map the space** $\mathbb{V} = \left\{ \left(\frac{\rho}{N}, \dots, \frac{\rho}{N}, \frac{\tau}{M}, \dots, \frac{\tau}{M}, \tau, S, \rho \right) \mid (\tau, S, \rho) \in \mathbb{R}^3 \right\}$ **into itself**

Physical interpretation at particular point in the moduli space

$$\vec{v} = (\vec{a}^{(1)}, \ldots, \vec{a}^{(M)}, \tau_1, \ldots, \tau_{M-1}, \tau, S, \rho)^T \longrightarrow \Big(\underbrace{\boxed{\tfrac{\rho}{N}, \ldots, \tfrac{\rho}{N}}}_{M(N-1)\text{-times}}, \underbrace{\boxed{\tfrac{\tau}{M}, \ldots, \tfrac{\tau}{M}}}_{M\text{-times}}, \tau, S, \rho \Big)^T$$

Identify all gauge parameters of all gauge nodes

Identify all couplings

Remaining parameters can be arranged in the period matrix of a genus 2 surface

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$$\mathfrak{t}_6 : (\tau, S, \rho) \longrightarrow (\tau, -S, \rho)$$

$$m^{dd} = \begin{pmatrix} 1 & m_{\mathfrak{ab}} & \infty \\ m_{\mathfrak{ab}} & 1 & \infty \\ \infty & \infty & 1 \end{pmatrix}$$

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with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generators of $PSL(2,\mathbb{Z}) \subset Sp(4,\mathbb{Z})$

Quiver Algebras

Consider a quiver Γ as a collection of nodes and (oriented) edges

Quiver Algebras

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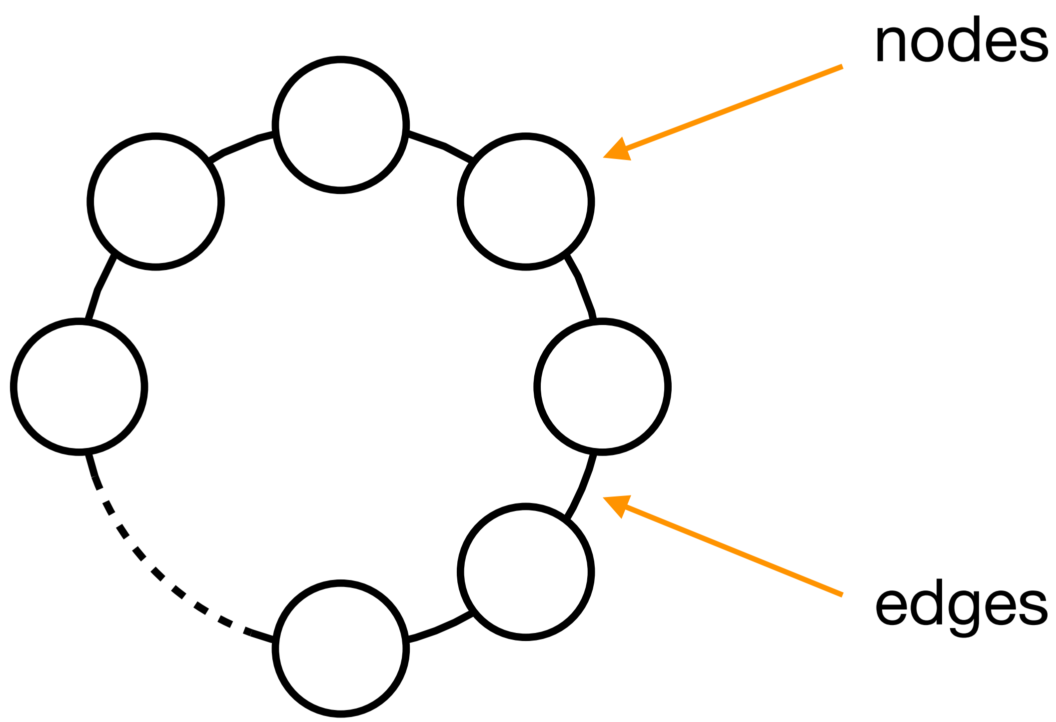

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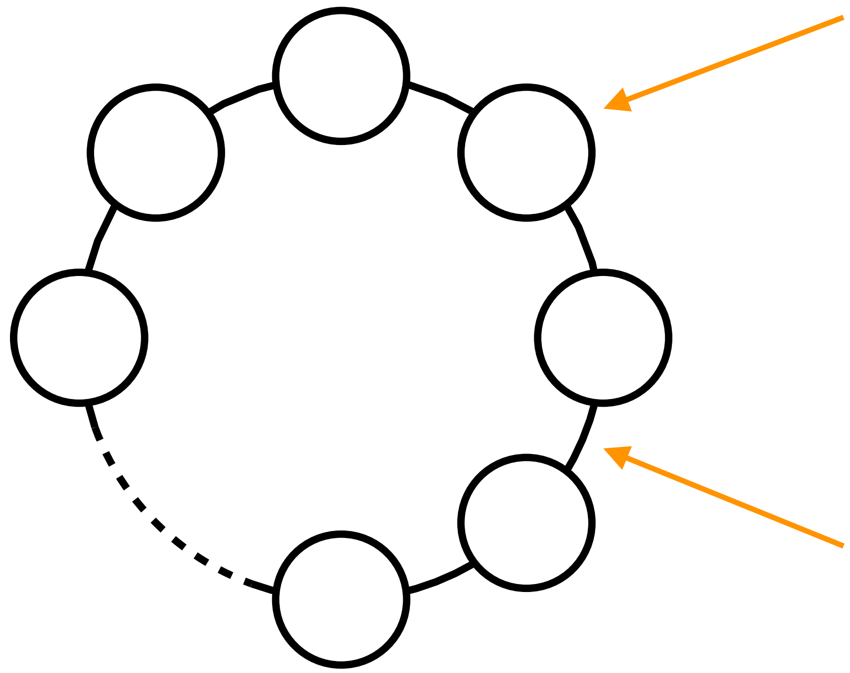

Γ itself encodes an algebraic structure, e.g.: Cartan matrix

[Kimura, Pestun 2015, 2016,2017]

$$c_{ij}^{[m]} = (1 + (t/q)^m)\delta_{ij} - \sum_{e:i \to j} \mu_e^m (t/q)^m - \sum_{e:j \to i} \mu_e^{-m}$$

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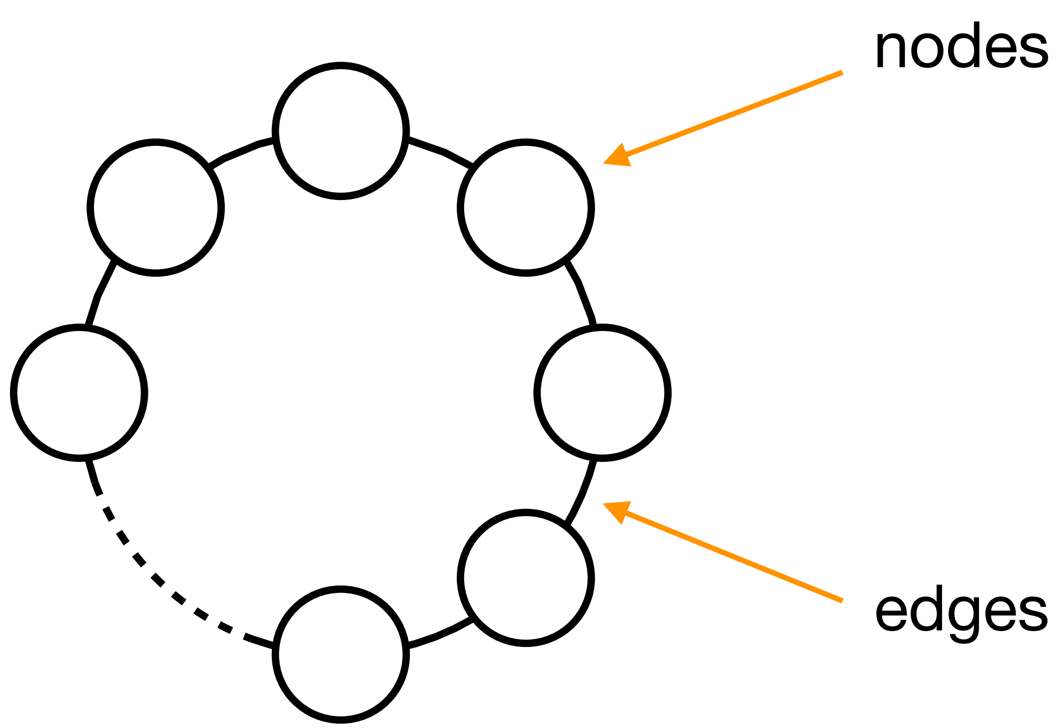

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Notation:

$\mu_e \cdots$ mass deformation of edge e

$q = e^{2\pi i \epsilon_1}$

$t = e^{-2\pi i \epsilon_2}$

Quiver Algebras

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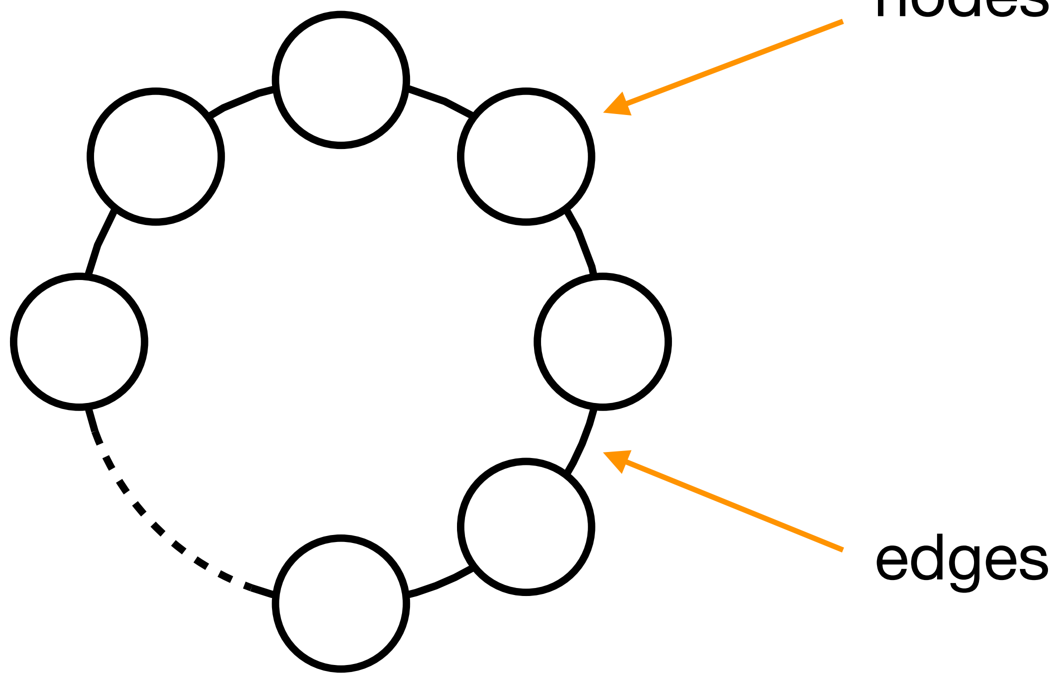

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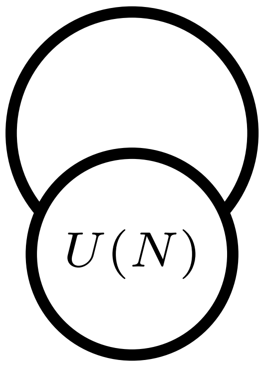
independent of $\epsilon_{1,2}$ for $\epsilon_1 = -\epsilon_2 = \epsilon$ and symmetric under $\mu_e \leftrightarrow \mu_e^{-1}$

$$c_{ij}^{[m]} = 2\delta_{ij} - \sum_{e:i\to j} \mu_e^m - \sum_{e:j\to i} \mu_e^{-m}$$

M=1 Gauge Theory Partition Function

Simples Case: $U(N)$ **single node**



$$c^{[n]} = [1 + (t/q)^n - (t/q)^n Q_S^n - Q_S^{-n}] \xrightarrow{q=t} [2 - Q_S^n - Q_S^{-n}]$$

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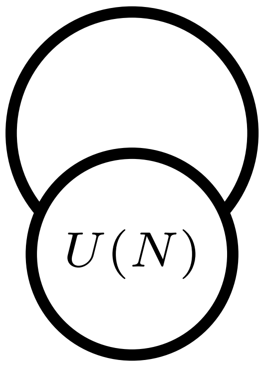

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[SH, Iqbal, Rey 2015]
[SH, Filoche 2022]

$$\mathcal{Z}_{N,1} = W_N(\emptyset) \sum_{\alpha_1,\ldots,\alpha_N} Q_\tau^{|\alpha_1|+\ldots+|\alpha_N|} \left(\prod_{k=1}^{N} \frac{\vartheta_{\alpha_k\alpha_k}(Q_S;\rho)}{\vartheta_{\alpha_k\alpha_k}(1;\rho)} \right) \prod_{1\le i<j\le N} \mathcal{T}_{\alpha_j\alpha_i}(\rho, S, \widehat{a}^{(1)}_{1,\ldots,N-1}, \epsilon)$$

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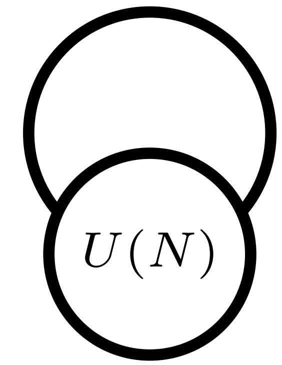

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non-perturbative
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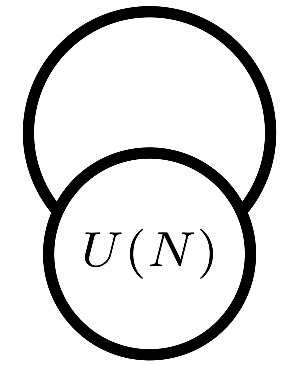

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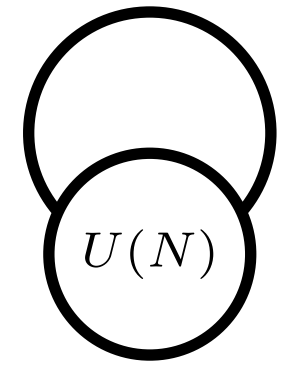

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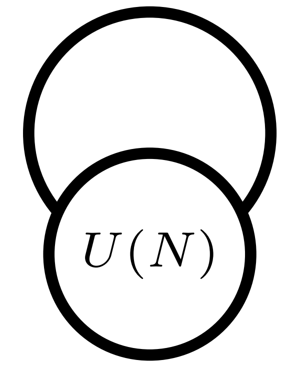

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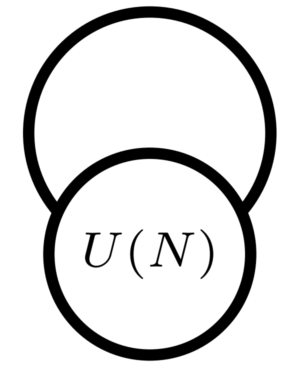

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Explicit Form:

[SH, Filoche 2022]

$$\mathcal{T}_{\alpha_j\alpha_i} = \left(-\frac{\phi_{-2}(S;\rho)}{4\pi^2}\right)^{|\alpha_i|+|\alpha_j|} \prod_{\kappa=\pm1} \left[\left(\prod_{(r,s)\in\alpha_j} \Omega\left(\mathfrak{a}_{ij} + \epsilon\, n_{r,s}^{\alpha_j,\alpha_i}, \kappa\, S;\rho\right) \right) \left(\prod_{(r,s)\in\alpha_i} \Omega\left(\mathfrak{a}_{ij} - \epsilon\, n_{r,s}^{\alpha_i,\alpha_j}, \kappa\, S;\rho\right) \right) \right]$$

M=1 Gauge Theory Partition Function

Simples Case: $U(N)$

$$c^{[n]} = [1 + (t/q$$

Partition Function:

$$\mathcal{Z}_{N,1} = \boxed{W_N(\emptyset)}$$

α_1

non-perturba
contribution

Notation:

Theta functions:

$$\vartheta_{\mu\nu}(x;\rho) = \prod_{(i,j)\in\mu} \vartheta\left(x^{-1} q^{-\nu_j^t+i-\frac{1}{2}} t^{-\mu_i+j-\frac{1}{2}};\rho\right) \prod_{(i,j)\in\nu} \vartheta\left(x^{-1} q^{\mu_j^t-i+\frac{1}{2}} t^{\nu_i-j+\frac{1}{2}};\rho\right)$$

$$\vartheta(x;\rho) = \left(x^{1/2} - x^{-1/2}\right) \prod_{k=1}^{\infty} (1 - xQ_\rho^k)(1 - x^{-1}Q_\rho^k) = \frac{iQ_\rho^{-1/8}\theta_1(z;\rho)}{\prod_{k=1}^{\infty}(1-Q_\rho^k)}$$

Standard Jacobi Forms

$$\phi_{-2}(\rho,z) = \frac{\theta_1^2(z;\rho)}{\eta^6(\rho)} \qquad \text{and} \qquad \phi_0(\rho,z) = 8\sum_{a=2}^{4} \frac{\theta_a^2(z;\rho)}{\theta_a^2(0,\rho)}$$

Kronecker-Eisenstein Series

$$\Omega(u,v;\rho) = \exp\left(2\pi i v \frac{\mathrm{Im}(u)}{\mathrm{Im}(\rho)}\right) \frac{\theta_1(u+v;\rho)\theta_1'(0;\rho)}{\theta_1(u;\rho)\theta_1(v;\rho)} \qquad \forall u,v \in \mathbb{C}$$

Explicit Form:

[SH, Filoche 2022]

$$\mathcal{T}_{\alpha_j\alpha_i} = \left(-\frac{\phi_{-2}(S;\rho)}{4\pi^2}\right)^{|\alpha_i|+|\alpha_j|} \prod_{\kappa=\pm1} \left[\left(\prod_{(r,s)\in\alpha_j} \Omega\left(\mathfrak{a}_{ij} + \epsilon\, n_{r,s}^{\alpha_j,\alpha_i}, \kappa\, S;\rho\right)\right)\left(\prod_{(r,s)\in\alpha_i} \Omega\left(\mathfrak{a}_{ij} - \epsilon\, n_{r,s}^{\alpha_i,\alpha_j}, \kappa\, S;\rho\right)\right)\right]$$

M=1 Gauge Theory Partition Function

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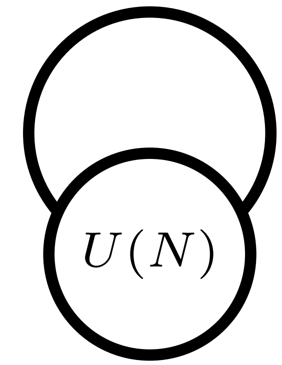

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Contribution of Nekrasov Subfunction can be written as correlation function

$$Q_\tau^{\sum_{i=1}^N |\alpha_i|} \prod_{1\le i<j\le N} \mathcal{T}_{\alpha_j\alpha_i} = \left\langle 0 \middle| \prod_{x\in\mathcal{X}_{\alpha_1,\ldots\alpha_N}}^{\prec} S_x \middle| 0 \right\rangle \Big/ \left\langle 0 \middle| \prod_{x\in\mathcal{X}_{\emptyset,\ldots\emptyset}}^{\prec} S_x \middle| 0 \right\rangle$$

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Chern characters in moduli space of instanton

$$\mathcal{T}_{\alpha_j\alpha_i} = \left(-\frac{\phi_{-2}(S;\rho)}{4\pi^2}\right)^{|\alpha_i|+|\alpha_j|} \prod_{\kappa=\pm 1}\left[\left(\prod_{(r,s)\in\alpha_j}\Omega\left(\mathfrak{a}_{ij}+\epsilon\, n_{r,s}^{\alpha_j,\alpha_i},\kappa\, S;\rho\right)\right)\left(\prod_{(r,s)\in\alpha_i}\Omega\left(\mathfrak{a}_{ij}-\epsilon\, n_{r,s}^{\alpha_i,\alpha_j},\kappa\, S;\rho\right)\right)\right]$$

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$$Q_\tau^{\sum_{i=1}^N|\alpha_i|}\prod_{1\le i<j\le N}\mathcal{T}_{\alpha_j\alpha_i} = \left\langle 0\left|\prod_{x\in\mathcal{X}_{\alpha_1,\ldots\alpha_N}}^{\prec} S_x\right|0\right\rangle \Big/ \left\langle 0\left|\prod_{x\in\mathcal{X}_{\emptyset,\ldots\emptyset}}^{\prec} S_x\right|0\right\rangle$$

[Kimura, Pestun 2016]

Chern characters in moduli space of instanton

Screening currents:

$$S_x = :\exp\left(s_0\log x+\tilde{s}_0+\sum_{m\in\mathbb{Z}^*}s_m^{(+)}x^{-m}+s_m^{(-)}x^m\right):$$

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Cartan matrix not inert under non-perturbative symmetries, thus non-trivial action on quiver algebra

Examples of Action:

[Filoche, SH, Kimura 2023]

Shift of mass parameter

$$(\widehat{a}_1^{(1)}, \widehat{a}_2^{(1)} \ldots, \widehat{a}_{N-1}^{(1)}, \rho, S, \tau) \longrightarrow (\widehat{a}_1^{(1)}, \widehat{a}_2^{(1)} \ldots, \widehat{a}_{N-1}^{(1)}, \rho, \rho - S, \tau - 2NS + N\rho)$$

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$$\mathcal{Z}_{N,1}=W_N(\emptyset)\sum_{\alpha_1,\ldots,\alpha_N}Q_\tau^{|\alpha_1|+\ldots+|\alpha_N|}\left(\prod_{k=1}^N\frac{\vartheta_{\alpha_k\alpha_k}(Q_S;\rho)}{\vartheta_{\alpha_k\alpha_k}(1;\rho)}\right)\prod_{1\leq i<j\leq N}\mathcal{T}_{\alpha_j\alpha_i}(\rho,S,\widehat{a}^{(1)}_{1,\ldots,N-1},\epsilon)$$

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Algebra deformation

$$c^{[m]}\longrightarrow 2-(Q_S^{-m}Q_\rho^m+Q_S^mQ_\rho^{-m})$$

$$\left[s_m^{(\pm)},s_{m'}^{(\pm)}\right]\longrightarrow\left[s_m^{(\pm)},s_{m'}^{(\pm)}\right]\pm\frac{(Q_\rho^{\mp m}Q_S^{\pm m}-Q_S^{\mp m})}{m}\delta_{m+m',0}$$

Summary and Conclusions

Studied dualities in a class of Little String Orbifolds:

* partition function $\mathcal{Z}_{N,M}$ compute as topological string partition function on $X_{N,M}$
* weak coupling regions give rise to different (but equivalent) expansions of $\mathcal{Z}_{N,M}$ that can be interpreted as instanton partition functions, dualities:

$$[U(M)]^N \Longleftrightarrow [U(M')]^{N'} \quad \text{for} \quad \begin{array}{l} NM = N'M' \\ \mathrm{gcd}(N,M) = \mathrm{gcd}(N',M') \end{array}$$

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Future directions:

* extension to more general quiver and gauge groups
* generalisation beyond free energy and partition function
* extension to further (phenomenologically realistic) theories