Automorphic Lie Algebras on Complex Tori

Casper Oelen

Heriot-Watt University

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Overview of talk

- What are automorphic Lie algebras?
- Motivation/history
- Classification of automorphic Lie algebras on complex tori[1]

[1] Vincent Knibbeler, Sara Lombardo, and Casper Oelen. "Automorphic Lie algebras on complex tori". In: *Proceedings of the Edinburgh Mathematical Society (to appear)* (2024).

What are automorphic Lie algebras?

Let X be a Riemann surface and $\mathfrak{g}$ be a complex finite-dimensional, semisimple Lie algebra. Automorphic Lie algebras (aLias) are defined as Lie algebras of meromorphic maps

$$X \to \mathfrak{g},$$

with the following properties:

1. Lie structure $[f, g](p) = [f(p), g(p)]$, $p \in X$
2. holomorphic outside a set of punctures
3. equivariant with respect to a group Γ acting on X and $\mathfrak{g}$ by automorphisms

- Automorphic Lie algebras generalise various Lie algebras
  - (twisted) loop algebras, (twisted) current algebras, Onsager algebras
- Appear in integrable systems (e.g. reduction of Lax pairs)
- Applications in *geometric deep learning*[2]

[2] Vincent Knibbeler. "Computing equivariant matrices on homogeneous spaces for geometric deep learning and automorphic Lie algebras". In: *Advances in Computational Mathematics* 50.2 (2024), p. 27.

The ingredients of an automorphic Lie algebra are:

- Finite-dimensional complex Lie algebra $\mathfrak{g}$
- Riemann surface X
- Discrete group Γ acting on X and $\mathfrak{g}$ via faithful homomorphisms

$$\rho : \Gamma \to \mathrm{Aut}(\mathfrak{g}), \quad \sigma : \Gamma \to \mathrm{Aut}(X)$$

- The algebra $\mathcal{O}_{\mathbb{X}}$ of meromorphic functions on X holomorphic on $\mathbb{X} := X \setminus S$, with $\sigma(\Gamma)S \subset S$

Alternative definition aLias

An aLia is a fixed point Lie subalgebra of $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}}$ with respect to the action $\gamma \cdot (A \otimes f(z)) = \rho(\gamma)A \otimes f(\sigma(\gamma)^{-1}z)$, $\gamma \in \Gamma$:

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}})^{\rho\otimes\sigma(\Gamma)} = \{a \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{X}} : \gamma \cdot a = a \text{ for any } \gamma \in \Gamma\}.$$

Twisted loop algebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathbb{C}[z, z^{-1}]$ the space of Laurent polynomials. Let ρ be an order n automorphism of $\mathfrak{g}$.

- Form the *Loop algebra* $\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ with bracket $[A \otimes f, B \otimes g] := [A, B] \otimes fg$.
- $\mathcal{L}(\mathfrak{g})$ is the Lie algebra of Laurent polynomials $f : \mathbb{C} \setminus \{0\} \to \mathfrak{g}$.
- The *twisted* loop algebra $\mathcal{L}(\mathfrak{g}, \rho)$ is the space of equivariant maps $f : \mathbb{C} \setminus \{0\} \to \mathfrak{g}$:
$$\rho f(z) = f(\epsilon z), \quad \epsilon^n = 1.$$

Kac (1969) proved that for any inner automorphism ρ, there is an isomorphism

$$\mathcal{L}(\mathfrak{g}, \rho) \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$$

of $\mathbb{Z}$-graded Lie algebras. The isomorphism can be written as

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}])^{C_n} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]^{C_n} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}].$$

History/Motivation

The notion of reduction group was introduced by A.V. Mikhailov in the context of reduction of Lax pairs [Mik80].

- Certain "elliptic automorphic Lie algebras" appear in the works of Reiman and Semenov-Tyan-Shanskii [RSTS89] and Uglov [Ugl94]
- ALias as a subject on its own was introduced by Mikhailov and Lombardo in [LM04],[LM05]. Further work related to integrable systems by Bury and Mikhailov [BM21]
- Algebraic development by Knibbeler and Lombardo with Sanders [KLS20], [KLS17], with Veselov [KLV22] and with Oelen [KLO24]
- In algebra, aLias are known as equivariant map algebras, introduced by Neher, Savage and Senesi [NSS12]

The classification of aLias is part of the programme of classifying Lax operators and hence of classifying integrable systems.

Motivation for aLias with genus ≥ 1

Suppose one wants to construct a Lax pair $L(z), M(z) \in \mathfrak{g}$ on a curve of genus g. It is known that for $g \geq 1$, there is an obstruction as a consequence of the Riemann-Roch Theorem. Generically:

$$\text{number of equations} > \text{number of variables}$$

whenever $g \geq 1$. Possible ways to resolve this:

1. Tyurin parameters
2. Impose symmetry to obtain consistent system

Point 2 is related to aLias. We demand that $L(z), M(z)$ satisfy

$$\rho(\gamma)L(\sigma(\gamma)^{-1}z) = L(z), \quad \rho(\gamma)M(\sigma(\gamma)^{-1}z) = M(z),$$

where ρ, σ represent the actions of the reduction group Γ.

The goal

Classify

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\sigma(\Gamma)},$$

where $\rho : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ and $\sigma : \Gamma \to \mathrm{Aut}(T)$ are (faithful) homomorphisms. The programme is:

- Classify groups Γ that can be faithfully embedded in *both* $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(T)$
- Classify the embeddings ρ and σ
- Compute ring of invariants $\mathcal{O}_{\mathbb{T}}^{\Gamma}$

In certain cases we can construct an intertwining map that allows us to find explicit normal forms and classify aLias

$$(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{K},$$

where $K \lhd \Gamma$.

- Classify $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}/K})^{\Gamma/K}$

Different geometries

- The case of genus 0: Intensively studied over the past two decades [LS10], [KLS20]
- Hyperbolic: Automorphic Lie algebras of modular type: [KLV22]
- Flat geometry: Genus 1 case - the topic of this talk [KLO24]

The classification of $(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma}$

Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The following Lie algebras appear in the classification:

1. $$\mathfrak{C}_\tau = \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[x, y]/(y^2 - 4x^3 + g_2(\tau)x + g_3(\tau)),$$

   with Lie structure inherited from $\mathfrak{sl}_2(\mathbb{C})$.

2. $$\mathfrak{S}_\tau = \mathbb{C}\langle E, F, H\rangle \otimes_{\mathbb{C}} \mathbb{C}[x],$$

   with Lie structure (linear over $\mathbb{C}[x]$)

   $$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H \otimes (4x^3 - g_2(\tau)x - g_3(\tau)).$$

3. The Onsager algebra $\mathfrak{O}$

The Lie algebras that appear in the classification of

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(\Gamma)}$$

fall into three (pairwise non-isomorphic) classes determined by the branch points of the canonical projection

$$\pi : \mathbb{T} \to \mathbb{T}/\Gamma.$$

# branch points	Lie algebra
0	$\mathfrak{C}_{[\tau]}$
2	$\mathfrak{O}$
3	$\mathfrak{S}_{[\tau]}$

Table: Lie algebra associated to the number of branch points of the quotient map $\mathbb{T} \to \mathbb{T}/\Gamma$.

- $\dim \mathfrak{A}/[\mathfrak{A},\mathfrak{A}] = \#(\text{branch points of } \pi)$
- $\mathfrak{C}_{[\tau]} \cong \mathfrak{C}_{[\tau']} \iff [\tau] = [\tau']$
- $[\tau] = [\tau'] \implies \mathfrak{S}_{[\tau]} \cong \mathfrak{S}_{[\tau']}$, but $\mathfrak{S}_{[\tau]} \cong \mathfrak{S}_{[\tau']} \overset{?}{\implies} [\tau] = [\tau']$

Classification of Γ

The group $\mathrm{Aut}(T)$ of biholomorphic automorphisms of a torus T is

$$\mathrm{Aut}(T) = \mathrm{Aut}_0(T) \ltimes t(T),$$

where $t(T)$ = subgroup of translations of T and $\mathrm{Aut}_0(T)$ is the subgroup of automorphisms that fix 0.
Finite subgroups of $\mathrm{Aut}(T)$ are of the form

$$C_\ell \ltimes (C_N \times C_M),$$

for some $\ell \in \{1, 2, 3, 4, 6\}$ and $N, M \in \mathbb{Z}_{\geq 1}$. The finite groups that *may* embed simultaneously in $\mathrm{Aut}(\mathfrak{sl}_2)$ and $\mathrm{Aut}(T)$ are given by

$$1 \ltimes C_N, \quad C_2 \ltimes C_N \cong D_N, \quad C_3 \ltimes (C_2 \times C_2) \cong A_4, \quad C_\ell \ltimes 1,$$

using Klein's classification of finite subgroups of $\mathrm{Aut}(\mathfrak{sl}_2)$.

Classification of subgroups of $\mathrm{Aut}(T)$

The subgroups of $\mathrm{Aut}(T)$ which are isomorphic to one of the finite groups of our list are classified by the following list, up to conjugation.

1. $C_N = \langle r : r^N = 1 \rangle$,
    1. $C_\ell \subset \mathrm{Aut}_0(T)$, $\quad r(z) = e^{2\pi i/\ell} z \quad (\ell \in \{2, 3, 4, 6)\})$.
    2. $C_N \subset t(T)$, $\quad r(z) = z + \alpha \quad$ (α is a N-torsion point in T).
2. $D_N = \langle s, r : s^2 = r^N = 1, (sr)^2 = 1 \rangle$,
    1. $C_2 \times C_2 \subset t(T)$, $\quad s(z) = z + \tau/2, \quad r(z) = z + 1/2$.
    2. $C_2 \ltimes C_N \subset \mathrm{Aut}_0(T) \ltimes t(T)$, $\quad s(z) = -z, \quad r(z) = z + \alpha$ (α is a N-torsion point in T).
3. $A_4 = \langle s, r_1, r_2 : s^3 = r_1^2 = r_2^2 = 1, sr_1s^{-1} = r_1r_2 = r_2r_1, sr_2s^{-1} = r_1 \rangle$, $\tau = e^{2\pi i/3}$,

$$s(z) = e^{2\pi i/3} z, \quad r_1(z) = z + 1/2.$$

Automorphic functions

Algebras of automorphic functions $\mathcal{O}_{\mathbb{T}}^{\Gamma}$ play a prominent role in computing aLias. Let $T = \mathbb{C}/\Lambda$. For Γ in our list:

- $\mathcal{O}_{\mathbb{T}}^{C_N} = \mathbb{C}[\wp, \wp']$, where $g(T/C_N) = 1$
- $\mathcal{O}_{\mathbb{T}}^{C_\ell} = \begin{cases} \mathbb{C}[\wp], & \ell = 2 \\ \mathbb{C}[\wp'], & \ell = 3 \\ \mathbb{C}[\wp^2], & \ell = 4 \\ \mathbb{C}[\wp^3], & \ell = 6 \end{cases}$, where $g(T/C_\ell) = 0$
- $\mathcal{O}_{\mathbb{T}}^{C_2 \times C_2} = \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}]$
- $\mathcal{O}_{\mathbb{T}}^{D_N} = \mathbb{C}[\wp]$
- $\mathcal{O}_{\mathbb{T}}^{A_4} = \mathbb{C}[\wp'_{\frac{1}{2}\Lambda_{\omega_6}}]$

Example 1: Landau-Lifshitz equation

The aLia with $\Gamma = C_2 \times C_2$ plays a prominent role in integrable systems:

- Appears in Sklyanin's Lax pair for the Landau-Lifshitz equation

The Landau-Lifshitz equation

$$S_t = S \times S_{xx} + S \times JS,$$

can be written as the compatibility condition $[L, M] = 0$ where

$$L(z) = \partial_x - (iS_1 A_1 + S_2 A_2 + iS_3 A_3),$$
$$M(z) = \partial_t - \frac{1}{2}(iU_1 A_1 + U_2 A_2 + iU_3 A_3 + iS_1 A_1' + S_2 A_2' + iS_3 A_3'),$$

with $U = S \times S_x$. A_k, A_k' form a basis of $(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2 \times C_2}$ over $\mathbb{C}[\wp_{\frac{1}{2}\Lambda}]$:

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2 \times C_2} = \bigoplus_{k=1}^{3} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}] A_k \oplus \mathbb{C}[\wp_{\frac{1}{2}\Lambda}] A_k'$$

Example 2: Onsager's algebra

Onsager's algebra: Used in the solution of the 2D Ising model. $\mathfrak{O}$ is generated by A_k, G_m with brackets

$$\begin{aligned}[A_k, A_l] &= 4G_{k-l},\\ [A_k, G_m] &= 2(A_{k-m} - A_{k+m}),\\ [G_m, G_n] &= 0,\end{aligned}$$

with $G_{-m} = -G_m$ $(m > 0)$ and $G_0 = 0$.

Theorem (Knibbeler, Lombardo, Veselov (21'))

$$\mathfrak{O} \cong \mathbb{C}\langle h, e, f\rangle \otimes_{\mathbb{C}} \mathbb{C}[J]$$

with Lie structure

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \otimes J(J-1).$$

Elliptic realisation of Onsager's algebra

Theorem ((Genus 0 case) Knibbeler, Lombardo, O (24'))

Let $\rho : C_\ell \to \mathrm{Aut}(\mathfrak{sl}_2)$ *and* $\sigma : C_\ell \to \mathrm{Aut}(T)$ *be monomorphisms and assume that* $g(T/\sigma(C_\ell)) = 0$*. Then*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_\ell)} \cong \mathfrak{O},$$

if and only if $\ell \in \{3, 4, 6\}$.

For example, letting $\sigma(r)z = \omega_3 z$, we have

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\sigma(C_\ell)} = \mathbb{C}\langle e \otimes \wp, f \otimes \wp^2, h\rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp'],$$

where e, f, h is the standard basis of $\mathfrak{sl}_2$.

Notice

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_\ell} \ncong \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{C_\ell}$$

Constructing intertwiners

Suppose that we can find $\Psi : T \to \mathrm{Aut}_{\mathcal{O}_\mathbb{T}}(\mathfrak{g} \otimes \mathcal{O}_\mathbb{T})$ such that

1. Ψ is meromorphic on T and holomorphic on $T \setminus \Gamma \cdot \{0\}$
2. $\Psi(\sigma(\gamma)z) = \rho(\gamma)\Psi(z)$
3. $\det(\Psi(z)) = 1$

Then

$$\mathfrak{g} \otimes \mathcal{O}_\mathbb{T} = \Psi(z)(\mathfrak{g} \otimes \mathcal{O}_\mathbb{T}) = \Psi(z)(\mathfrak{g}) \otimes \mathcal{O}_\mathbb{T} \cong \mathfrak{g} \otimes \mathcal{O}_\mathbb{T}.$$

Hence

$$(\mathfrak{g} \otimes \mathcal{O}_\mathbb{T})^\Gamma \cong \mathfrak{g} \otimes \mathcal{O}_\mathbb{T}^\Gamma. \tag{1}$$

For this to apply, it is necessary that $\Gamma_p = 1$ for all $p \in T$ (i.e. $T \to T/\Gamma$ has no branch points). Indeed, if $\Gamma_p \neq 1$, then the evaluation representation

$$\mathrm{ev}_p : (\mathfrak{g} \otimes \mathcal{O}_\mathbb{T})^\Gamma \to \mathfrak{g}^{\Gamma_p}$$

evaluates to a lower dimensional Lie subalgebra of $\mathfrak{g}$.

Strategy for $\Gamma = C_N$: Look for a matrix $\Omega_j(z) \in SL_2(\mathbb{C})$ such that

$$\Omega_j(z+\alpha) = \begin{pmatrix} \omega_N^j & 0 \\ 0 & \omega_N^{-j} \end{pmatrix} \Omega_j(z),$$

where $\alpha \in T$ N-torsion point and $j \not\equiv N/2 \mod N$. Introduce

$$\Omega_j(z) = \begin{pmatrix} \varphi_{-j}(z) & \frac{1}{\mu}\varphi_j(z)\varphi_{-2j}(z) + \frac{\lambda}{2\mu}\varphi_{-j}(z) \\ \varphi_j(z) & \frac{1}{\mu}\varphi_{-j}(z)\varphi_{2j}(z) - \frac{\lambda}{2\mu}\varphi_j(z) \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{C}$ and

$$\varphi_j(z) = \sum_{k=0}^{N-1} \frac{\wp'(z-k\alpha)}{\omega_N^{kj}(\wp(z-k\alpha) - \wp(\alpha))}, \quad (\alpha \ N\text{-torsion point})$$

- $\wp$ is the Weierstrass p-function associated to the appropriate lattice
- φ_j $(j \neq 0)$ has simple poles in $\mathbb{Z}\alpha$ and $\varphi_j(z+\alpha) = \omega_N^{-j}\varphi_j(z)$
- Ω_j is a meromorphic map $T \to SL_2(\mathbb{C})$, holomorphic on $\mathbb{T}$
- $\Omega_j(z+\alpha) = \mathrm{diag}(\omega_N^j, \omega_N^{-j})\Omega_j(z)$

Theorem (Genus 1 case. Knibbeler, Lombardo, O (24'))

Let $\rho : C_N \to \mathrm{Aut}(\mathfrak{sl}_2)$ *and* $\sigma : C_N \to t(T)$ *be monomorphisms. Then*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_N)} \cong \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\tilde{\sigma}(C_N)}.$$

A normal form is given by

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_N)} = \mathbb{C}\langle E, F, H\rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp_\Lambda, \wp'_\Lambda],$$

where Λ *is a suitable lattice, and with brackets*

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H,$$

where E, F *and* H *are the images under* $\mathrm{Ad}(\Omega_j)$*, for some integer* $j \not\equiv 0 \mod N/2$*, of* $e \otimes 1$*,* $f \otimes 1$ *and* $h \otimes 1$*, respectively.*

An explicit basis for $(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_N)}$

Let

$$H_j = \mathrm{Ad}(\Omega_j(z)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_j = \mathrm{Ad}(\Omega_j(z)) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$F_j = \mathrm{Ad}(\Omega_j(z)) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$H_j = \frac{1}{\mu} \begin{pmatrix} \varphi_{-j}^2\varphi_{2j} + \varphi_{-2j}\varphi_j^2 & -2\varphi_{-j}\varphi_j\varphi_{-2j} - \lambda\varphi_{-j}^2 \\ 2\varphi_{-j}\varphi_j\varphi_{2j} - \lambda\varphi_j^2 & -\varphi_{-j}^2\varphi_{2j} - \varphi_{-2j}\varphi_j^2 \end{pmatrix},$$

$$E_j = \begin{pmatrix} -\varphi_{-j}\varphi_j & \varphi_{-j}^2 \\ -\varphi_j^2 & \varphi_{-j}\varphi_j \end{pmatrix},$$

and

$$F_j = \frac{1}{4\mu^2} \begin{pmatrix} 4\varphi_{-j}\varphi_j\varphi_{-2j}\varphi_{2j} + \lambda^2\varphi_{-j}\varphi_j + 2\lambda\mu & -4\varphi_j^2\varphi_{-2j}^2 - 4\lambda\varphi_{-j}\varphi_j\varphi_{-2j} - \lambda^2\varphi_{-j}^2 \\ 4\varphi_{-j}^2\varphi_{2j}^2 - 4\lambda\varphi_{-j}\varphi_j\varphi_{2j} + \lambda^2\varphi_j^2 & -4\varphi_j\varphi_{-j}\varphi_{2j}\varphi_{-2j} - \lambda^2\varphi_{-j}\varphi_j - 2\lambda\mu \end{pmatrix}.$$

Higher dimensional base Lie algebras

For certain representations $C_N \to \mathrm{Aut}(\mathfrak{sl}_n)$, we can obtain intertwiners in a simple way, using the construction for $n = 2$. For example, for n odd we define

$$\tilde{\Omega}(z) = \mathrm{diag}(1, \Omega_{j_1}(z), \dots, \Omega_{j_m}(z)).$$

We still have $\det(\tilde{\Omega}(z)) = 1$ and

$$\tilde{\Omega}(z + \alpha) = R\tilde{\Omega}(z),$$

where $\alpha \in T$ is a N-torsion point and $R = 1 \oplus R_{j_1} \oplus \dots \oplus R_{j_m}$. Hence

$$(\mathfrak{sl}_n \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_N} \cong \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{C_N}.$$

$\Gamma = D_N$

Corollary (Knibbeler, Lombardo, O (24'))

Let $\rho : D_N \to \mathrm{Aut}(\mathfrak{sl}_2)$ *and* $\sigma : D_N \to \mathrm{Aut}(T)$ *be monomorphisms. Then*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho \otimes \tilde{\sigma}(D_N)} \cong \mathfrak{S}_\tau,$$

for some $\tau \in \mathbb{H}$. *A normal form is given by*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho \otimes \tilde{\sigma}(D_N)} = \mathbb{C}\langle \tilde{E}, \tilde{F}, \tilde{H} \rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp_\Lambda],$$

where Λ *is a suitable lattice and* $\tilde{E} = E \otimes \wp'_\Lambda$, $\tilde{F} = F \otimes \wp'_\Lambda$ *and* $\tilde{H} = H$. *The Lie structure is given by*

$$[\tilde{H}, \tilde{E}] = 2\tilde{E}, \quad [\tilde{H}, \tilde{F}] = -2\tilde{F}, \quad [\tilde{E}, \tilde{F}] = \tilde{H} \otimes (4\wp_\Lambda^3 - g_2\wp_\Lambda - g_3). \quad (2)$$

$\Gamma = C_2 \times C_2$ setup

Any representation $\rho : C_2 \times C_2 \to \mathrm{Aut}(\mathfrak{sl}_2)$ is equivalent to

$$\rho(r_1) = \mathrm{Ad}(T_1), \quad \rho(r_2) = \mathrm{Ad}(T_2),$$

where $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Suppose we look for matrix valued function $\Omega : \mathbb{T} \to GL_2(\mathbb{C})$ which is $C_2 \times C_2$-equivariant:

$$\Omega(z + \tfrac{1}{2}) = T_1 \Omega(z), \quad \Omega(z + \tfrac{\tau}{2}) = T_2 \Omega(z),$$

for all $z \in \mathbb{T}$. There is an obstruction. One way around this: Look instead for

$$\Omega : \mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{C}),$$

$$\Omega(z + \tfrac{1}{2}) = T_1 \Omega(z), \quad \Omega(z + \tfrac{\tau}{2}) = f(z) T_2 \Omega(z),$$

where $f : \mathbb{C} \to \mathbb{C}$ is some function.

$\Gamma = C_2 \times C_2$ setup

Let $\tau \in \mathbb{H}$ and define the theta functions with characteristics

$$\theta_{a,b}(z|\tau) = \sum_{k\in\mathbb{Z}} \exp\left\{\pi i\tau(k+a)^2 + 2\pi i(k+a)(z+b)\right\},$$

where $a, b \in \mathbb{Q}$. This defines a holomorphic, quasi-periodic function on $\mathbb{C}$:

$$\theta_{a,b}(z+1|\tau) = e^{2\pi i a}\theta_{a,b}(z|\tau), \quad \theta_{a,b}(z+\tau|\tau) = e^{-\pi i(2z+2b+\tau)}\theta_{a,b}(z|\tau).$$

The Jacobi Theta functions are defined by

$$\theta_1(z|\tau) = -\theta_{\frac{1}{2},\frac{1}{2}}(z|\tau), \qquad \theta_3(z|\tau) = \theta_{0,0}(z|\tau),$$
$$\theta_2(z|\tau) = \theta_{\frac{1}{2},0}(z|\tau), \qquad \theta_4(z|\tau) = \theta_{0,\frac{1}{2}}(z|\tau).$$

$\Gamma = C_2 \times C_2$ setup

Define

$$\Omega(z) = \begin{pmatrix} \theta_3(2z|2\tau) & \left(\theta_3^2(0|2\tau)\frac{\theta_4(2z|2\tau)}{\theta_1(2z|2\tau)} + \theta_2^2(0|2\tau)\frac{\theta_1(2z|2\tau)}{\theta_4(2z|2\tau)}\right)\theta_2(2z|2\tau) \\ \theta_2(2z|2\tau) & \left(\theta_3^2(0|2\tau)\frac{\theta_1(2z|2\tau)}{\theta_4(2z|2\tau)} + \theta_2^2(0|2\tau)\frac{\theta_4(2z|2\tau)}{\theta_1(2z|2\tau)}\right)\theta_3(2z|2\tau) \end{pmatrix}.$$

Then

1. $\Omega(z+\frac{1}{2}) = T_1\Omega(z)$
2. $\Omega(z+\frac{\tau}{2}) = e^{-\pi i(2z+\frac{\tau}{2})}T_2\Omega(z)$
3. $\det\Omega(z) = -\theta_2^2(0|\tau)\theta_1(2z|\tau)$

Proposition

For Ω as above:

$$\mathrm{Ad}(\Omega) \in \mathrm{Aut}_{\mathcal{O}_{\mathbb{T}}}(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})$$

Denote by h, e, f the standard basis of $\mathfrak{sl}_2$.

Theorem (Knibbeler, Lombardo, O (24'))

Let $\rho : C_2 \times C_2 \to \mathrm{Aut}(\mathfrak{sl}_2)$ *and* $\sigma : C_2 \times C_2 \to \mathrm{Aut}(T)$ *be monomorphisms and assume that* $g(T/\sigma(C_2 \times C_2)) = 1$. *Then*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_2\times C_2)} \cong \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\tilde{\sigma}(C_2\times C_2)}.$$

A normal form is given by

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho\otimes\tilde{\sigma}(C_2\times C_2)} = \mathbb{C}\langle H, E, F\rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp_{\frac{1}{2}\Lambda}, \wp'_{\frac{1}{2}\Lambda}]$$

with

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H,$$

where $H = \mathrm{Ad}(\Omega)h$, $E = \mathrm{Ad}(\Omega)e$, $F = \mathrm{Ad}(\Omega)f$.

The explicit generators of $(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{C_2 \times C_2}$

Write θ_j for $\theta_j(0|\tau)$, $j = 1, 2, 3, 4$.

$$H(z) = -\frac{1}{\theta_2^2} \begin{pmatrix} -\frac{\theta_2^4}{\theta_3\theta_4}\lambda_3(z)\lambda_4(z) & -\theta_2\psi_-(z)\lambda_2(z) \\ \theta_2\psi_+(z)\lambda_2(z) & \frac{\theta_2^4}{\theta_3\theta_4}\lambda_3(z)\lambda_4(z) \end{pmatrix},$$

$$E(z) = -\frac{1}{2\theta_2^2} \begin{pmatrix} -\theta_2\lambda_2(z) & \theta_3\lambda_3(z) + \theta_4\lambda_4(z) \\ -\theta_3\lambda_3(z) + \theta_4\lambda_4(z) & \theta_2\lambda_2(z) \end{pmatrix},$$

$$F(z) = -\frac{1}{2\theta_2^2} \begin{pmatrix} \theta_2\psi_-(z)\psi_+(z)\lambda_2(z) & -\psi_-(z)\left(\theta_3\lambda_3(z) - \theta_4\lambda_4(z)\right) \\ \psi_+(z)\left(\theta_3\lambda_3(z) + \theta_4\lambda_4(z)\right) & -\theta_2\psi_-(z)\psi_+(z)\lambda_2(z) \end{pmatrix}$$

where $\psi_\pm = \pm\frac{\theta_4^2}{\theta_3}\lambda_3(z) - \frac{\theta_3^2}{\theta_4}\lambda_4(z)$ and $\lambda_j(z) = \frac{\theta_j(2z|\tau)}{\theta_1(2z|\tau)}$.

$\Gamma = A_4$

Theorem (Knibbeler, Lombardo, O (24'))

Let $T \cong T_{\omega_6}$ *and* $\rho : A_4 \to \mathrm{Aut}(\mathfrak{sl}_2)$ *and* $\sigma : A_4 \to \mathrm{Aut}(T)$ *be monomorphisms. Let* $\mathbb{T} = T \setminus A_4 \cdot \{0\}$. *There is the following isomorphism of Lie algebras:*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho \otimes \tilde{\sigma}(A_4)} \cong \mathfrak{O},$$

where $\mathfrak{O}$ *is the Onsager algebra. A normal form is given by*

$$(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\rho \otimes \tilde{\sigma}(A_4)} = \mathbb{C}\langle E \otimes \wp_{\frac{1}{2}\Lambda_{\omega_6}}, F \otimes \wp^2_{\frac{1}{2}\Lambda_{\omega_6}}, H \rangle \otimes_{\mathbb{C}} \mathbb{C}[\wp'_{\frac{1}{2}\Lambda_{\omega_6}}],$$

where E, F, H *are the generators of the* $C_2 \times C_2$*-aLia.*

A summary

Our classification of $(\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^\Gamma$ can be summarised in the following table:

	0	2	3
C_2	$\mathfrak{C}_\tau$		$\mathfrak{S}_\tau$
$C_N,\ N = 3, 4, 6$	$\mathfrak{C}_\tau$	$\mathfrak{O}$	
$C_N,\ N \neq 2, 3, 4, 6$	$\mathfrak{C}_\tau$		
$C_2 \times C_2$	$\mathfrak{C}_\tau$		$\mathfrak{S}_\tau$
$D_N,\ N \geq 3$			$\mathfrak{S}_\tau$
A_4		$\mathfrak{O}$	

Table: Isomorphism classes of aLias for each symmetry group Γ and each number of branch points of the quotient map $\mathbb{T} \to \mathbb{T}/\Gamma$.

The Lie algebras $\mathfrak{S}_\tau$: Some remarks

For $\mathfrak{C}_\tau$ we had $\mathfrak{C}_\tau \cong \mathfrak{C}_{\tau'} \iff T_\tau \cong T_{\tau'}$. For $\mathfrak{S}_\tau$ we know that

$$T_\tau \cong T_{\tau'} \implies \mathfrak{S}_\tau \cong \mathfrak{S}_{\tau'}.$$

It remains a conjecture that the other direction holds as well. The question boils down to studying the Lie algebras

$$\mathfrak{S}_{\lambda(\tau)} := (\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[x])_{\lambda(\tau)}$$

with Lie structure

$$[H, E] = 2E, \quad [H, F] = -2F$$

and

$$[E, F]_{\lambda(\tau)} := H \otimes x(x-1)(x-\lambda(\tau)),$$

where λ is the modular Lambda function.

Outlook

ALias (genus 1 case) $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma}$ have been classified for $\mathfrak{g} = \mathfrak{sl}_2$. Current research related to this is focused on

1. Extending the classification to $\mathfrak{g}$ semisimple and $\Gamma = 1 \ltimes (C_N \times C_M) \subset \mathrm{Aut}(T)$
2. Proving or disproving $(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}})^{\Gamma} \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{T}}^{\Gamma}$ whenever $\Gamma \hookrightarrow \mathrm{Aut}(T)$ is embedded without fixed points
3. Applying in the context of elliptic Lax pairs

Thank you for listening!

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