Bi-Hamiltonian geometry of WDVV equations: general results

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Integrable systems and automorphic forms

Joint ongoing work with Raffaele Vitolo

Witten–Dijkgraaf–Verlinde–Verlinde equations

The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $F_{1\alpha\beta} := \dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix
2. $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} F_{\epsilon\alpha\beta}$ structure constants of an associative algebra
3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity ($d_1 = 1$)

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

The system of WDVV equations follows,

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta} F_{\mu\gamma\nu} - F_{\lambda\alpha\nu} F_{\mu\beta\gamma}) = 0. \qquad \text{(WDVV)}$$

The WDVV system is invariant under linear change of transformations that preserves t^1,

$$\tilde{t}^i = c^i_j t^j \quad \text{with} \quad c^i_1 = \delta^i_1.$$

With quasihomogeneity, if quasihomogeneity weights are distinct, the matrix $\eta_{\alpha\beta}$ can be reduced [Dubrovin, 1994] to

$$\begin{pmatrix} \mu & 0 & & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ & & \ddots & & \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix}$$

Symmetries of WDVV

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Without quasihomogeneity, e.g. for $n = 3$, there are 4 canonical matrices [Mokhov, Pavlenko, 2018],

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$\lambda^2 = 1, \qquad \lambda^2 = 1, \qquad \qquad \lambda^2 = 1,\ \mu^2 = 1$$

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

How many independent equations are in WDVV system?

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

How many independent equations are in WDVV system?

$$\begin{aligned}
&\mu f_{yyz}(f_{zzz} - f_{yzz}) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,\\
&f_{xxy}f_{yzz} - f_{xxz}f_{yyz} - \mu f_{zzz}f_{xyz} + f_{zzz} + f_{xyy}f_{xzz} + \mu f_{xzz}f_{yzz} - f_{xyz}^2 = 0,\\
&f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + \mu f_{yyz}f_{xzz} - \mu f_{xyz}f_{yzz} + f_{yzz} = 0,\\
&f_{xxy}f_{xzz} - \mu f_{xxz}f_{zzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} + \mu f_{xzz}^2 = 0,\\
&f_{xxz}f_{xyy} + \mu f_{xxz}f_{yzz} - f_{yyz}f_{xxx} - \mu f_{xzz}f_{xyz} + f_{xzz} = 0,\\
&f_{xxy}f_{xyy} + \mu f_{xxz}f_{yyz} - f_{xxx}f_{yyy} - \mu f_{xyz}^2 + 2f_{xyz} = 0.
\end{aligned}$$

for Dubrovin normal form

$$\eta^{(2)} = \begin{pmatrix} \mu & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

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How many independent equations are in WDVV system?

First of all, $F_{1\alpha\beta} = \eta_{\alpha\beta}$ completely specifies the dependence of F on t^1,

$$F = \frac{1}{6}\eta_{11}(t^1)^3 + \frac{1}{2}\sum_{k>1}\eta_{1k}t^k(t^1)^2 + \frac{1}{2}\sum_{k,s>1}\eta_{sk}t^st^kt^1 + f(t^2,\dots,t^N).$$

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Secondly, there are two apparent symmetries, $S_{\alpha\beta\gamma\nu} = S_{\gamma\nu\alpha\beta}$ and $S_{\alpha\nu\gamma\beta} = -S_{\alpha\beta\gamma\nu}$.

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

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In particular, any combination of parameters containing 3 or 4 identical letters gives rise to a trivial equation, and any equation with 2 identical letters can be brought to a form $S_{\alpha\alpha\gamma\nu}$.

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In particular, any combination of parameters containing 3 or 4 identical letters gives rise to a trivial equation, and any equation with 2 identical letters can be brought to a form $S_{\alpha\alpha\gamma\nu}$.

Lastly, choosing $\alpha \in \{2, \dots, N\}$ we can find a subsystem of the above system that is linear with respect to α-independent derivatives and solve it. The remaining equations then vanish.

Running example, WDVV with $N = 4$

Consider the WDVV system,

$$\begin{aligned}
&\mu f_{yyz}(f_{zzz} - f_{yzz}) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,\\
&f_{xxy}f_{yzz} - f_{xxz}f_{yyz} - \mu f_{zzz}f_{xyz} + f_{zzz} + f_{xyy}f_{xzz} + \mu f_{xzz}f_{yzz} - f_{xyz}^2 = 0,\\
&f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + \mu f_{yyz}f_{xzz} - \mu f_{xyz}f_{yzz} + f_{yzz} = 0,\\
&f_{xxy}f_{xzz} - \mu f_{xxz}f_{zzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} + \mu f_{xzz}^2 = 0,\\
&f_{xxz}f_{xyy} + \mu f_{xxz}f_{yzz} - f_{yyz}f_{xxx} - \mu f_{xzz}f_{xyz} + f_{xzz} = 0,\\
&f_{xxy}f_{xyy} + \mu f_{xxz}f_{yyz} - f_{xxx}f_{yyy} - \mu f_{xyz}^2 + 2f_{xyz} = 0.
\end{aligned} \tag{1}$$

for Dubrovin normal form

$$\eta^{(2)} = \begin{pmatrix} \mu & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Choose a variable x and see that the latter 5 equations are linear wrt f_{yyy}, f_{yyz}, f_{yzz}, f_{zzz}.

Mokhov–Ferapontov trick

1. Choose one distinguished independent variable t^k, $k > 1$, and all third-order derivatives f_σ such as $\sigma_k > 0$; introduce new variables $u^i = f_{(3,0,\ldots,0)}$, $u^2 = f_{(2,1,0,\ldots 0)}$, ..., $u^n = f_{(1,0,\ldots,2)}$, $n = N(N-1)/2$.

2. Choose another independent variable $t^h \neq t^k$, $h > 1$ and find $u^i_{t^h}$ as the t^k-derivative of an expression V^i:

$$u^i_{t^h} = V^i(\mathbf{u})_{t^k}. \tag{2}$$

   There are two possibilities:

   1. either $V^i(\mathbf{u})$ is one of the coordinates u^j, with $j \neq k$;
   2. V^i is a third-order derivative of f which is not one of the u^j. In this case, V^i must be expressed by means of one of the equations of the WDVV system. This is always possible due to the structure of the WDVV system.

Geometric picture

$$\{S_{\alpha\beta\gamma\nu}\} \subset J_3 E = (t^\lambda, f_\sigma).$$

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Given $J_mE = (x^\lambda, v^i_\sigma)$, consider the jet bundle J_1J_mE with coordinates $(x^\lambda, v^i_\sigma, \bar{v}^i_{\mu\tau})$ where σ, $\tau \in \mathbb{N}^N$ are multiindices such that $|\sigma|$, $|\tau| \leq m$, and μ is an index. Note that in cases when $\sigma = \mu + \tau$ the coordinates v^i_σ and $\bar{v}^i_{\mu\tau}$ are in general different.

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Then, the sesquiholonomic jet bundle $\hat{J}_{m+1}E \subset J_1J_mE$ is identified in coordinates as the subspace

$$v^i_\sigma = \bar{v}^i_{\mu\tau},$$

for all σ, $\tau \in \mathbb{N}^N$, $|\sigma| \leq m$, $\mu \in \{1, \ldots, N\}$, such that $\sigma = \mu + \tau$. So, the sesquiholonomic jet bundle can be endowed with coordinates $(x^\lambda, v^i_\sigma, v^i_{\mu\tau})$ where $|\sigma| \leq m$, $|\tau| = m$, and μ is an index.

Running example

Let us introduce new letters for third-order derivatives:

$$u^1 = f_{xxx},\ u^2 = f_{xxy},\ u^3 = f_{xxz},\ u^4 = f_{xyy},\ u^5 = f_{xyz},\ u^6 = f_{xzz},$$
$$u^7 = f_{yyy},\ u^8 = f_{yyz},\ u^9 = f_{yzz},\ u^{10} = f_{zzz}.$$

We have the following compatibility relations:

$$\begin{array}{lll}
u^1_y = u^2_x & u^1_z = u^3_x & u^2_z = u^3_y \\
u^2_y = u^4_x & u^2_z = u^5_x & u^4_z = u^5_y \\
u^3_y = u^5_x & u^3_z = u^6_x & u^5_z = u^6_y \\
u^4_y = u^7_x & u^4_z = u^8_x & u^7_z = u^8_y \\
u^5_y = u^8_x & u^5_z = u^9_x & u^8_z = u^9_y \\
u^6_y = u^9_x & u^6_z = u^{10}_x & u^9_z = u^{10}_y
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u^4_y = u^7_x & u^4_z = u^8_x & u^7_z = u^8_y \\
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u^6_y = u^9_x & u^6_z = u^{10}_x & u^9_z = u^{10}_y
\end{array}$$

In general, a WDVV system in dimension N is equivalent to $N-2$ commuting hydrodynamic-type systems.

Consider two systems

$$u^i_y = (V^i)_x, \qquad u^i_z = (W^i)_x.$$

Two such systems are said to commute if and only if the Jacobi bracket of the right-hand sides vanishes: $[V, W] = 0$, where $V = (V^i)_x \partial_{u^i}$ and $W = (W^i)_x \partial_{u^i}$. This is equivalent to the requirement that W is a generalized symmetry of the system $u^i_y = (V^i)_x$ and vice versa.

Definition

We say that a quasilinear first-order system of conservation laws (2) where (u^i) are third-order derivatives of f and the equations are compatibility conditions for a WDVV system to be a *first-order WDVV system*.

Hamiltonian structures. Known results

$\underline{N=3}$
1st Dubrovin normal form ($\mu = 0$): local 3rd order + local 1st order [Ferapontov, Galvao, Mokhov, Nutku, 1997]
2nd Dubrovin normal form ($\mu \neq 0$): local 3rd order + nonlocal 1st order [Vašiček, Vitolo, 2021].
Mokhov–Pavlenko normal forms: local 3rd order + nonlocal 1st order [Vašiček, Vitolo, 2021].
All: local 3rd order + (non)local 1st order [Vašiček, Vitolo, 2021].

$\underline{N=4}$
1st Dubrovin normal form ($\mu = 0$): local 3rd order + local 1st order [Ferapontov, Mokhov, 1996], [Pavlov, Vitolo, 2015]
2nd Dubrovin normal form ($\mu \neq 0$): local 3rd order [Vašiček, Vitolo, 2021].

$\underline{N=5}$
Dubrovin normal forms ($\mu \in \{0,1\}$): local 3rd order [Vašiček, Vitolo, 2021].

Third-order homogeneous Hamiltonian operator in a canonical Doyle–Potemin form is

$$A_3^{ij} = \mathrm{D}_x \circ (h^{ij}\mathrm{D}_x + c_k^{ij} u_x^k) \circ \mathrm{D}_x.$$

Given $c_{ijk} = h_{iq}h_{jp}c_k^{pq}$, the skew-symmetry conditions and the Jacobi identities for the operator above are equivalent to

$$c_{skm} = \frac{1}{3}(h_{sm,k} - h_{sk,m}),$$
$$h_{mk,p} + h_{kp,m} + h_{mp,k} = 0$$
$$c_{msk,l} = -f^{pq}c_{pml}c_{qsk},$$

which implies that g_{ij} is a Monge metric,

$$g_{ij}\mathrm{d}u^i\mathrm{d}u^j = a_{ij}\mathrm{d}u^i\mathrm{d}u^j + b_{ijk}\mathrm{d}u^i(u^j\mathrm{d}u^k - u^k\mathrm{d}u^j) + c_{ijkl}(u^i\mathrm{d}u^j - u^j\mathrm{d}u^i)(u^k\mathrm{d}u^l - u^l\mathrm{d}u^k).$$

Operator has a projective-geometric nature [Ferapontov, Pavlov, Vitolo, 2014].

The metric f_{ij} can be factorized [Balandin, Potemin, 2001] as

$$f_{ij} = \phi_{\alpha\beta}\psi_i^\alpha\psi_j^\beta, \quad \Big(\text{or, in a matrix form,} \quad f = \Psi\Phi\Psi^\top\Big) \tag{3}$$

where ϕ is a constant non-degenerate symmetric matrix of dimension n, and

$$\psi_k^\gamma = \psi_{ks}^\gamma u^s + \omega_k^\gamma$$

is a non-degenerate square matrix of dimension n, with the constants ψ_{ij}^γ and ω_k^γ satisfying the relations

$$\begin{aligned} &\psi_{ij}^\gamma = -\psi_{ji}^\gamma, \\ &\phi_{\beta\gamma}(\psi_{il}^\beta\psi_{jk}^\gamma + \psi_{jl}^\beta\psi_{ki}^\gamma + \psi_{kl}^\beta\psi_{ij}^\gamma) = 0, \\ &\phi_{\beta\gamma}(\omega_i^\beta\psi_{jk}^\gamma + \omega_j^\beta\psi_{ki}^\gamma + \omega_k^\beta\psi_{ij}^\gamma) = 0. \end{aligned}$$

For the conservative system $\mathbf{u}_t = (V(\mathbf{u}))_x$, the necessary and sufficient conditions to admit the above Hamiltonian operator are

$$\begin{aligned} &h_{im}V_j^m = h_{jm}V_i^m, \\ &V_{ij}^k = h^{ks}c_{smj}V_i^m + h^{ks}c_{smi}V_j^m. \end{aligned}$$

Running example, 3rd order Hamiltonian operator

$$
\begin{aligned}
&h_{11} = u_4^2, \quad h_{12} = (\mu u_5 - 2)u_5, \quad h_{13} = 2u_4(1 - \mu u_5),\\
&h_{14} = \mu u_3 u_5 - u_1 u_4 - u_3, \quad h_{15} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 - u_3 u_4 - u_6) + u_2,\\
&h_{16} = (\mu u_5 - 1)^2, \quad h_{22} = 2u_3(\mu u_5 - 1),\\
&h_{23} = -\mu^2 u_5 u_6 - \mu(u_2 u_5 + u_3 u_4 - u_6) + u_2, \quad h_{24} = \mu u_3^2,\\
&h_{25} = -\mu^2 u_3 u_6 - \mu(u_1 u_5 + u_2 u_3) + u_1, \quad h_{26} = 2\mu u_3(\mu u_5 - 1),\\
&h_{33} = \mu^2(2u_4 u_6 + u_5^2) + 2\mu(u_2 u_4 - u_5) + 2,\\
&h_{34} = -\mu^2 u_3 u_6 + \mu(u_1 u_5 - u_2 u_3) - u_1, \quad h_{35} = \mu((\mu u_6 + u_2)^2 - h_{14}),\\
&h_{36} = \mu h_{23}, \quad h_{44} = u_1^2, \quad h_{45} = -2\mu u_1 u_3,\\
&h_{46} = \mu^2 u_3^2, \quad h_{55} = \mu^2(2u_1 u_6 + u_3^2) + 2\mu u_1 u_2,\\
&h_{56} = \mu h_{25}, \quad h_{66} = 2\mu^2 u_3(u_5 \mu - 1).
\end{aligned}
$$

First-order Hamiltonian operator

Standard Dubrovin–Novikov Hamiltonian operator is of the form

$$A_1^{ij} = g^{ij} \mathrm{D}_x + \Gamma^{ij}_k u^k_x, \quad \text{where} \quad \Gamma^{ij}_k = -g^{is} \Gamma^j_{sk}.$$

The operator A_1 is Hamiltonian if and only if the following conditions hold:

$$g^{ij} = g^{ji},$$
$$g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k,$$
$$g^{is} \Gamma^{jk}_s = g^{js} \Gamma^{ik}_s,$$
$$R^{ij}_{hl} = 0.$$

Thus, g is a flat metric, and Γ^i_{jk} are the Christoffel symbols of the corresponding Levi-Civita connection.

First nonlocal generalisation of the standard Dubrovin–Novikov Hamiltonian operator is of the form

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c u^i_x \mathrm{D}_x^{-1} u^j_x, \quad \text{where} \quad \Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}.$$

The operator A_1 is Hamiltonian if and only if the following conditions hold:

$$g^{ij} = g^{ji},$$
$$g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k,$$
$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s,$$
$$R^{ij}_{hl} = c.$$

Thus, g is a constant-curvature metric, and Γ^i_{jk} are the Christoffel symbols of the corresponding Levi-Civita connection.

Second nonlocal generalisation of the standard Dubrovin–Novikov Hamiltonian operator is of the form

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + w^i_k(\mathbf{u})u^k_x \mathrm{D}_x^{-1} \circ w^j_h(\mathbf{u})u^h_x.$$

The operator A_1 is Hamiltonian if and only if the following conditions hold:

$$g^{ij} = g^{ji}, \quad g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k, \quad g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s,$$
$$g_{ik}w^k_j = g_{jk}w^k_i,$$
$$\nabla_k w^i_j = \nabla_j w^i_k,$$
$$R^{ij}_{hl} = w^i_l w^j_h - w^i_h w^j_l.$$

Thus, the metric g and the affinor w satisfy the Gauss–Peterson–Codazzi equations for hypersurfaces M^n in a pseudo-Euclidean space E^{n+1}, that is, the metric g_{ij} plays the role of the first quadratic form of M^n, and the affinor w^i_j is the Weingarten operator.

First-order Hamiltonian operator

Second nonlocal generalisation of the standard Dubrovin–Novikov Hamiltonian operator is of the form

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + \sum_{\alpha,\beta} c^{\alpha\beta} w^i_{\alpha k} u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h} u^h_x,$$

where $(c^{\alpha\beta})$ is a real symmetric matrix

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$$g^{ij} = g^{ji},$$
$$g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k,$$
$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s,$$
$$g_{ik} w^k_{\alpha j} = g_{jk} w^k_{\alpha i},$$
$$\nabla_k w^i_{\alpha j} = \nabla_j w^i_{\alpha k},$$
$$[w_\alpha, w_\beta] = 0,$$
$$R^{ij}_{hl} = c^{\alpha\beta}\left(w^i_{\alpha l} w^j_{\beta h} - w^i_{\alpha h} w^j_{\beta l}\right).$$

Tricks to simplify the computation of A_1

Conjecture

Let the system of conservation laws admit a third-order Hamiltonian operator as above parameterised by a Monge metric h with Monge decomposition

$$h = \Psi\Phi\Psi^\top,$$

where Φ is a constant matrix, and the entries of Ψ are linear in u_k's. Then the metric g defining the compatible Ferapontov-type first-order Hamiltonian operator is of the form

$$g = \Psi^{-1}Q(\Psi^{-1})^\top, \quad (g^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta),$$

where Q is a matrix whose entries are polynomials in u_k of order at most 2.

Valid for many known examples [Opanasenko, Vitolo, 2024].

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When computing with first-order homogeneous Hamiltonian operators it turns out that it is more natural to use contravariant quantities. In particular, we will use the contravariant version of the Riemannian curvature:

$$R^{ijk}_l = g^{is}g^{jt}R^k_{tsl} = g^{is}(\partial_l\Gamma^{jk}_s - \partial_s\Gamma^{jk}_l) + \Gamma^{ij}_s\Gamma^{sk}_l - \Gamma^{sj}_l\Gamma^{ik}_s.$$

For quasilinear systems:

$$g^{ik}V^j_k = g^{jk}V^i_k, \quad \nabla^i V^j_k = \nabla^j V^i_k, \quad [V, w_\alpha] = 0.$$

$$\nabla^i V^j_k - \nabla^j V^i_k = 0,$$

In view of the fact that the system is written in a conservative form, this condition simplifies to $V^s_k \Gamma^{ji}_s - V^j_s \Gamma^{si}_k = 0$, where $\Gamma^{ij}_k = -g^{is}\Gamma^j_{sk}$. It further simplifies to

$$\Gamma^{sij} V^k_s - \Gamma^{skj} V^i_s = 0, \quad \text{where} \quad \Gamma^{ijk} = g^{is}\Gamma^{jk}_i.$$

The simplification is related to the fact that in order to find Γ^{ijk} one only needs to use the higher-indices tensor g^{ij},

$$\Gamma^{lij} = \frac{1}{2}\left(g^{is}g^{jl}_{,s} + g^{ls}g^{ij}_{,s} - g^{js}g^{il}_{,s}\right),$$

which in applications happens to be much simpler than its lower-indices counterpart.

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Lastly, to find constants $c^{\alpha\beta}$ we check the condition

$$R^{sij}{}_l = g^{ks}c^{\alpha\beta}(w^j_{\alpha k}w^i_{\beta l} - w^i_{\alpha k}w^j_{\beta l}),$$

Running example, first-order Hamiltonian operator

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + \sum_{\alpha,\beta=0}^{3} c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$

where

$$(g^{ij}) = (\Psi^{-1}) Q (\Psi^{-1})^\top,$$

Φ is a constant matrix, and the entries of Ψ are linear in u_k's,

$$\Psi = \begin{pmatrix} \frac{u_4}{\mu} & \frac{u_5}{\mu} & 1 & 0 & 0 & 0 \\ 0 & \frac{u_3}{\mu} & 0 & -u_5 & 1 & 0 \\ -u_5 & -\frac{u_2}{\mu} - u_6 & 0 & u_4 & 0 & 1 \\ -\frac{u_1}{\mu} & 0 & 0 & -u_3 & 0 & 0 \\ u_3 & -\frac{u_1}{\mu} & 0 & \mu u_6 + u_2 & 0 & 0 \\ 0 & u_3 & 0 & -\mu u_5 + 1 & 0 & 0 \end{pmatrix},$$

$$Q^{11} = -\frac{4}{\mu}u_3u_5 + \frac{4}{\mu^2}u_1u_4 + u_6^2, \quad Q^{12} = -\frac{2}{\mu}u_3u_6 + \frac{4}{\mu^2}u_1u_5,$$

$$Q^{13} = u_1u_5 - \frac{1}{\mu}u_3u_6 + u_2u_3 + \frac{2}{\mu}u_1, \quad Q^{14} = -\frac{2}{\mu}(u_2u_5 - u_4u_3 + u_6),$$

$$Q^{15} = -\mu u_5u_6 + u_2u_5 + u_3u_4 + u_6, \quad Q^{16} = \mu u_6^2 + 2u_3u_5, \quad Q^{22} = \frac{2}{\mu^2}(u_1u_6 - u_3^2),$$

$$Q^{23} = -\frac{2}{\mu}u_1u_2 + u_3^2, \quad Q^{24} = \frac{4}{\mu}u_3u_5 - \frac{2}{\mu}u_2u_6 - u_6^2,$$

$$Q^{25} = u_3u_5 - \frac{1}{\mu}u_1u_4 - \frac{2}{\mu}u_3 - \frac{1}{\mu}u_2^2, \quad Q^{26} = -\frac{1}{\mu}u_1u_5 + u_3u_6 - \frac{1}{\mu}u_2u_3,$$

$$Q^{33} = \mu^2u_3^2 - 2\mu u_1u_2, \quad Q^{34} = -\mu u_3u_5 + u_1u_4 + u_2^2 + 4u_3,$$

$$Q^{35} = \mu^2u_3u_5 - \mu u_1u_4 - \mu u_2^2 - \mu u_3, \quad Q^{36} = \mu^2u_3u_6 - \mu u_1u_5 - \mu u_2u_3 + u_1,$$

$$Q^{44} = 2u_4u_6 - 2u_5^2, \quad Q^{45} = -\mu u_5^2 + 2u_2u_4 + 4u_5,$$

$$Q^{46} = -\mu u_5u_6 + u_2u_5 + u_3u_4 + 3u_6, \quad Q^{55} = \mu^2u_5^2 - 2\mu u_2u_4 - 2\mu u_5 - 2,$$

$$Q^{56} = \mu^2u_5u_6 - \mu u_2u_5 - \mu u_3u_4 - \mu u_6 + u_2, \quad Q^{66} = \mu^2u_6^2 - 2\mu u_3u_5 + 2u_3,$$

while the nonlocal part is defined by the matrix

$$\begin{pmatrix} c^{11} & c^{12} & c^{13} \\ c^{21} & c^{22} & c^{23} \\ c^{31} & c^{32} & c^{33} \end{pmatrix} = \begin{pmatrix} 0 & -\mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}.$$

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$
$$A_3^{ij} = \mathrm{D}_x \circ (h^{ij}\mathrm{D}_x + c^{ij}_k u^k_x) \circ \mathrm{D}_x.$$

Theorem

Let $u^i_{t^h} = (V^i)_{t^k}$ be a family of commuting first-order WDVV systems, $h = 2, \ldots, N$, $h \neq k$. If there is one value of h such that the first-order system is bi-Hamiltonian with a pair of compatible Hamiltonian operators A_1, A_3, then all first-order WDVV systems corresponding to all other values h are endowed with exactly the same bi-Hamiltonian pair.

Proof

It is known for the first-order operator [Ferapontov, 1995].

$$A_1^{ij} = g^{ij}\mathrm{D}_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta} w^i_{\alpha k}(\mathbf{u}) u^k_x \mathrm{D}_x^{-1} \circ w^j_{\beta h}(\mathbf{u}) u^h_x,$$
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$$h_{im} V^m_j = h_{jm} V^m_i, \quad V^k_{ij} = h^{ks} c_{smj} V^m_i + h^{ks} c_{smi} V^m_j.$$

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Compatibility of the operators A_1 and A_3 gives

$$h_{im} w^m_{\alpha j} = h_{jm} w^m_{\alpha i}, \quad w^k_{\alpha i,j} = h^{ks} c_{smj} w^m_{\alpha i} + h^{ks} c_{smi} w^m_{\alpha j}.$$

Theorem

An invariance transformation of the WDVV equation preserves the form of the Hamiltonian operators in a bi-Hamiltonian first-order WDVV system.

Proof

The symmetry group of a third-order WDVV projects to the symmetry group $\mathrm{GL}(N-1, \mathbb{C})$ of a first-order WDVV.

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Lemma: invariance transformations that involve only two independent variables preserve the form of the Hamiltonian operators [Vašiček, Vitolo, 2021].

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Lemma: invariance transformations that involve only two independent variables preserve the form of the Hamiltonian operators [Vašiček, Vitolo, 2021].

Any matrix in $\mathrm{GL}(\mathbb{C}^{N-1})$ can be generated by means of Gauss' elementary matrices (up to permutations).

Future research

- General "simplified" formula for WDVV system
- Show that any first-order WDVV system is bi-Hamiltonian.

Merci beaucoup!