Higher Bessel Functions: 1 Year Later

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joint with Ilia Gaiur and Duco van Straten (arXiv:2405.03015)

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Timeline

Clausen duplication (1828)

$$\left({}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix} \mid x\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a & a+b & 2b \\ & a+b+\frac{1}{2} & 2b \end{matrix} \mid x\right]$$

Schläfli formula (1876)

$${}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{nx}{2}\right] {}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{ny}{2}\right] = \int_{S^{n-1}} {}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{n}{2}(x+y+2\sqrt{xy} < e_1, \mu >)\right] d\mu$$

Sonine-Gegenbauer formula (1880; 1884)

$$J_0(x)J_0(y) = \frac{1}{2\pi} \int_{x-y}^{x+y} \frac{J_0(z)zdz}{\sqrt{x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2}}, \; x < y \in \mathbb{R}$$

Kontsevich (2007)

$$(\partial_t \circ f(t) \circ \partial_t + t)\,\phi = \lambda\phi, \quad f(t) = t^3 + At^2 + t$$

$$\phi_\lambda(x)\phi_\lambda(y) = \int \frac{\phi_\lambda(z)}{\sqrt{P(x,y,z)}}dz, \quad P(x,y,z) = \mathrm{discr}_t\,[f(t) - (t-x)(t-y)(t-z)]$$

Exhibit the motivic nature of multiplication kernels on the very explicit example of the kernels for these N-Bessel functions

i.e. write these kernels as periods of some algebraic families

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$N-$Bessel or $\mathbb{P}^{N-1}-$Quantum DO:

$$\boxed{\left(x\frac{d}{dx}\right)^N \Psi - \lambda x \Psi = 0}$$

Analytic solution

$$\Phi_N(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!^N} =$$

$$= \frac{1}{(2\pi i)^{N-1}} \oint \exp\left(\sum_{i=1}^{N-1} Y_i + \frac{x}{Y_1 Y_2 \dots Y_{N-1}}\right) \frac{dY_1}{Y_1}\frac{dY_2}{Y_2}\dots\frac{dY_{N_1}}{Y_{N-1}}$$

Multiplication Kernels:

$$\boxed{\Phi_N(\lambda, x)\Phi_N(\lambda, y) = \oint K_N(x, y|z)\Phi_N(\lambda, z)\frac{dz}{z}}$$

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Object of study

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And more general

$$\Phi_N(x_1)\Phi_N(x_2)\ldots\Phi_N(x_m) = \oint K_N^{(m)}(x_1, x_2, \ldots x_m|z)\Phi_N(z)\frac{dz}{z}$$

1. Previous Year reminder

2. *N*-Bessel kernels as periods

3. Singularities: Landau Discriminants, Buchstaber-Rees

4. Integrality properties: computer experiments

5. Connections

Previous Year reminder

Duplication kernels

Consider a "diagonal" $x = y$ in order to get one dimensional families

Convolution of power series and explicit numbers

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Convolution of power series and explicit numbers

Integral transformation to Sym^2 of DE

$$\Phi_N(x)^2 = \frac{1}{2\pi i}\oint K_N(x/z)\Phi_N(z)\frac{dz}{z}$$

For $N = 2$ – Clausen duplication

$$\Phi_2(x)^2 = \frac{1}{2\pi i}\oint_T \frac{1}{\sqrt{1-4x/z}}\Phi_2(z)\frac{dz}{z}.$$

which leads to

$$(\theta_x^3 - 4x\,\theta_x - 2x)\Phi_2(x)^2 = 0, \quad \theta_x = \frac{xd}{dx}$$

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Duplication kernels via multiplicative convolution

Taking the "diagonal" $y = x$ we obtain the Clausen duplication formula

$$\Phi_N(x)^2 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!^N}\right)^2 = \sum_{n=0}^{\infty} \frac{c_n}{n!^N} x^n, \quad c_n = \sum_{k=0}^{n} \binom{n}{k}^N$$

Hadamard product: $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$

$$(f * g)(x) = \frac{1}{2\pi i} \oint f(x/t) g(t) \frac{dt}{t} = \sum_{j=0}^{\infty} (a_j \cdot b_j) x^j.$$

Then

$$\Phi_N(x)^2 = \Phi_N(x) * \sum_{n=0}^{\infty} c_n x^n = \Phi_N(x) * K(x) = \frac{1}{2\pi i} \oint K(x/t) \Phi_N(t) dt/t$$

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Binomial coefficient sums

$$K_N(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k}^N \right] t^n, \quad t = x/z$$

$N = 2$ - Clausen duplication for Bessel

$$K_2(x) = \frac{1}{\sqrt{1-4t}}$$

$N = 3$ - Apéry like sequence of type A (D. Zagier). Heun function

$$t(t+1)(8t-1)K^{''} + (24t^2 + 14t - 1)K^{'} + (8t+2)K = 0$$

Generic $N > 1$: period from Landau-Ginzburg model

$$V_N = \prod_{i=1}^{N-1}(1+Y_i) + \prod_{i=1}^{N-1}(1+Y_i^{-1})$$

Corresponding family and its period

$$V_N = 1/t, \quad K_N(t) = \frac{1}{(2\pi i)^{N-1}} \oint_{T^{N-1}} \frac{1}{1 - tV_N} \frac{dY_1}{Y_1} \frac{dY_2}{Y_2} \cdots \frac{dY_{N-1}}{Y_{N-1}}$$

Picard-Fuchs operators

$N=2$ $\quad (4t-1)\,\theta_t+2t,$

$N=3$ $\quad (t+1)\,(8t-1)\theta_t^2+t\,(16t+7)\,\theta_t+2t(4t+1),$

$N=4$ $\quad (16t-1)(4t+1)\theta_t^3+6t(32t+3)\theta_t^2+2t(94t+5)\theta_t+2t(30t+1),$

$N=5$ $\quad (32t-1)(4t-7)^2(t^2-11t-1)\theta_t^4+2t(4t-7)(256t^3-2084t^2+4942t+143)\theta_t^3+$
$t(3072t^4-23024t^3+72568t^2-102261t-1638)\theta_t^2+$
$t(2048t^4-12896t^3+30072t^2-66094t-637)\theta_t+$
$2t(256t^4-1472t^3+1904t^2-7868t-49),$

$N=6$ $\quad (t-1)(27t+1)(64t-1)(75t^3+1420t^2+561t+9)\theta_t^6+$
$+(842400t^6+18022725t^5-1363487t^4-4622791t^3-127551t^2-1977t-9)\theta_t^5+$
$+5t(452880t^5+10962507t^4+2491544t^3-1779376t^2-46584t-168)\theta_t^4+$
$+5t(644760t^5+17135271t^4+6994741t^3-1716533t^2-56024t-96)\theta_t^3+$
$+2t(1282500t^5+36696915t^4+19164721t^3-2088858t^2-100236t-72)\theta_t^2+$
$+2t(540900t^5+16436910t^4+9826066t^3-428487t^2-38811t-9)\theta_t+$
$+12t^2(15750t^4+503175t^3+327205t^2-1845t-1044).$

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathrm{ord}\mathcal{D}_{\mathcal{N}}$	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40

Number of unipotent blocks increase on each fifth iteration

$$M_0^{(7)} \sim \begin{pmatrix} 1 & 1/2! & 1/3! & 1/4! & 1/5! & 1/6! & 0 & 0 \\ 0 & 1 & 1/2! & 1/3! & 1/4! & 1/5! & 0 & 0 \\ 0 & 0 & 1 & 1/2! & 1/3! & 1/4! & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2! & 1/3! & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/2! & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2! \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Mirror maps

$$N=3 \quad q-3q^2+3q^3+5q^4-18q^5+15q^6+24q^7-75q^8+57q^9+86q^{10}+O(q^{11})$$

$$N=4 \quad q-4q^2-6q^3+56q^4-45q^5-360q^6+894q^7+960q^8-6951q^9+4660q^{10}+O\left(q^{11}\right)$$

$$N=5 \quad q-5q^2-40q^3+115q^4-645q^5-12846q^6-177350q^7- \\ -2574585q^8-44198680q^9-736554815q^{10}+O\left(q^{11}\right)$$

$$N=6 \quad q-6q^2-135q^3-380q^4-24960q^5-696366q^6- \\ -26153302q^7-901888104q^8-35369115894q^9-1381135576280q^{10}+O\left(q^{11}\right)$$

$$N=7 \quad q-7q^2-371q^3-4543q^4-378637q^5-20096783q^6-1568975093q^7 \\ -112310305031q^8-9251250532328q^9-758736375700793q^{10}+O\left(q^{11}\right)$$

N-Bessel kernels as periods

$K_N(x, y|z)$ are periods of $\mathbb{P}^2$-families of algebraic varieties

$K_N^{(m)}(x_1, x_2 \dots x_m|z)$ are periods of $\mathbb{P}^m$-families

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Theorem

Let

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

Define

$$K_N(x,y,z) = \frac{1}{(2\pi i)^{N-1}}\oint \frac{1}{W_N(x,y,z|\mathbf{Y})}\prod_{j=1}^{N-1}\frac{dY_j}{Y_j}$$

Theorem (I. Gaiur, V.R., D. van Straten) $K_N(x,y,z)$ is a kernel for N-Bessel function

$$W_N(x,y,z|Y) = \prod_{j=1}^{N-1}(1+Y_j)\left(x + \frac{y}{Y_1Y_2\dots Y_{N-1}}\right) - z = 0$$

Proof: Kontsevich-Odesskii formal power series for kernel + combinatorics

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Multi-Kernels

Multi-product

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$K_N(x_1, x_2, \ldots x_m\,|z)$ is a convolution of $K_N(x, y|z)$ with itself $m-2$ times

Theorem (I. Gaiur, V.R., D. van Straten) Kernel $K_N(x_1, x_2, \ldots x_m\,|z)$ is a period of

$$W_N^{(m)}(x_1, \ldots x_m, z) = \prod_{j=1}^{N-1}\left(1 + \sum_{l=1}^{m-1} Y_j^{(l)}\right)\cdot\left(x_1 + \sum_{l=1}^{m-1}\prod_{j=1}^{N-1}\frac{x_{l+1}}{Y_j^{(l)}}\right) - z = 0,$$

i.e.

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Proof: Similar to the multiplication one

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i.e.

$$K_N(x_1,x_2,\dots x_m\,|z) = \frac{1}{(2\pi i)^{N-1}}\oint \frac{1}{W_N^{(m)}(x_1,x_2,\dots x_m,z)}\prod_{j,l}\frac{d\,Y_j^{(l)}}{Y_j^{(l)}}$$

Proof: Similar to the multiplication one

Multi-Kernels

Multi-product

$$\Phi_N(x_1)\Phi_N(x_2)\dots\Phi_N(x_{m-1})\Phi_N(x_m) = \frac{1}{2\pi i}\oint K_N(x_1, x_2, \dots x_m\,|z)\,\Phi_N(z)\frac{dz}{z}.$$

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In Watson (13.46, formula (9))

$$\int_0^\infty \prod_{i=1}^{4} J_0(a_n\lambda)\lambda d\lambda = \frac{1}{\pi^2}\left\{ \begin{array}{ll} \frac{1}{\Delta} K\left(\frac{\sqrt{a_1a_2a_3a_4}}{\Delta}\right), & \left|\frac{\sqrt{a_1a_2a_3a_4}}{\Delta}\right| < 1 \\ \frac{1}{\sqrt{a_1a_2a_3a_4}} K\left(\frac{\Delta}{\sqrt{a_1a_2a_3a_4}}\right), & \end{array}\right.$$

$K(k)$ is complete elliptic integral of the first kind

$$16\Delta^2 = \prod_{n=1}^{4}(a_1 + a_2 + a_3 + a_4 - 2a_n).$$

LHS is a "pairing"

$$\langle J_0(a_4\lambda), J_0(a_1\lambda)J_0(a_2\lambda)J_0(a_3\lambda)\rangle$$

RHS is a product kernel for 3 Bessel functions. Corresponding elliptic curve is

$$(1 + X + Y)\left(a_1^2 + \frac{a_2^2}{X} + \frac{a_3^2}{Y}\right) - a_4^2 = 0$$

Related Examples: Watson formula

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```
sage: X,Y,a1,a2,a3,a4=QQ["X,Y,a1,a2,a3,a4"].gens()
sage: W = (1+X+Y)*(X*Y*a1^2+Y*a2^2+X*a3^2) - X*Y*a4^2;
sage: from sage.schemes.toric.weierstrass import WeierstrassForm
sage: from sage.schemes.toric.weierstrass import Discriminant
sage: factor(Discriminant(W, [X,Y]))
(-1/16) * (-a1 - a2 + a3 - a4) * (-a1 - a2 + a3 + a4) * (-a1 + a2 + a3 -
a4) * (-a1 + a2 + a3 + a4) * (a1 - a2 + a3 - a4) * (a1 - a2 + a3 + a4) *
(a1 + a2 + a3 - a4) * (a1 + a2 + a3 + a4) * a4^4 * a3^4 * a2^4 * a1^4
```

$N = 3$ Elliptic family

Family

$$xXY(1+X)(1+Y) + y(1+X)(1+Y) - zXY = 0,$$

Deformation of Beauville family of type IV

$$(\tilde{X}+\tilde{Y})(\tilde{Y}+\tilde{Z})(x\tilde{Z}+y\tilde{X}) + z\tilde{X}\tilde{Y}\tilde{Z} = 0$$

Kernel

$$K_3(x,y,z) = -\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;-\frac{27xyz}{(x+y-z)^3}\right)}{x+y-z}.$$

Gauss hypergeometry as expected

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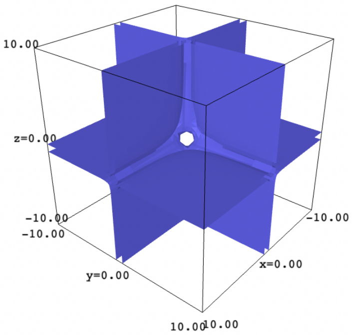
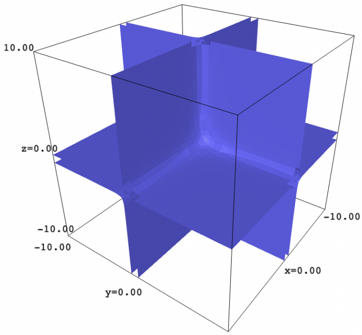


$[-2:-1:1]$



$[-2:1:1]$

Singularities: Landau Discriminants, Buchstaber-Rees

Appeared families have a very specific geometry of singularity loci

For standard multiplication formulas all singularities are of discriminantal type

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Product formulas singularities

The singularities of the family

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

is a projective curve in $\mathbb{P}^2$ which a union of triangle

$$xyz = 0$$

and irreducible rational curve

$$\Delta_N(x,y,z) = x^N + y^N + z^N + \ldots,$$

which is given by the property

$$\Delta_N(u^N, v^N, w^N) = \prod_{\omega,\eta}(u + \omega v + \eta w),$$

where in the product ω and η run over the N-th roots of unity.

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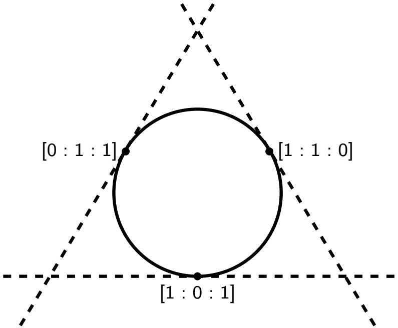
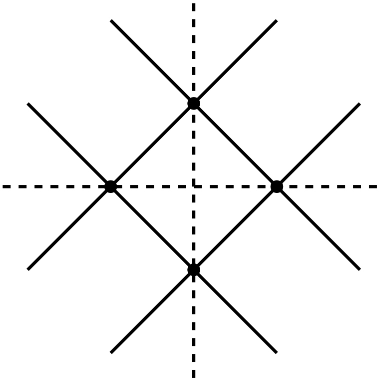
$N = 2$ singularity loci and unfolding





Theorem The polynomials $\Delta_N(x,y,z)$ may be expressed via T-discriminants of the $2N-2$ degree polynomials given by

$$P_{x,y,z}(T) = xT^{N-1}(1+T)^{N-1} + y(1+T)^{N-1} - T^{N-1}z.$$

More precisely, the equality holds

$$(-1)^N(N-1)^{2(N-1)}(xyz)^{N-2}\Delta_N(x,y,z) = \mathrm{disc}_T(P_{x,y,z}(T))$$

Theorem The projective curve $\Delta_N = 0$ is a rational curve that has $(N-1)(N-2)/2$ double points. All singularities belong to $\mathbb{P}^2(\mathbb{R})$. Moreover, coordinates of the singularities belong to $\mathbb{Q}(\cos(2\pi/N))$.

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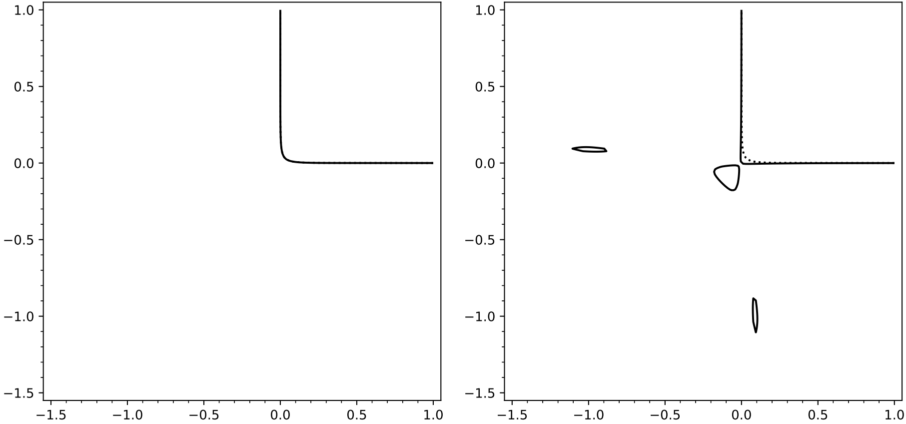
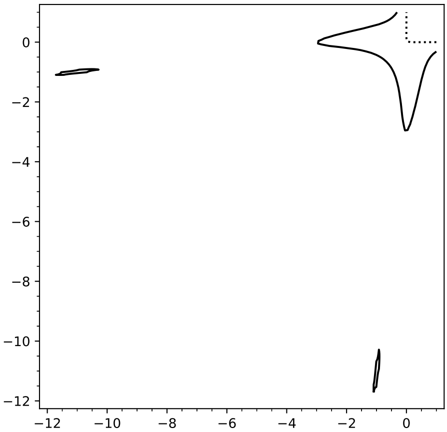
$N = 5\ \Delta = \epsilon$

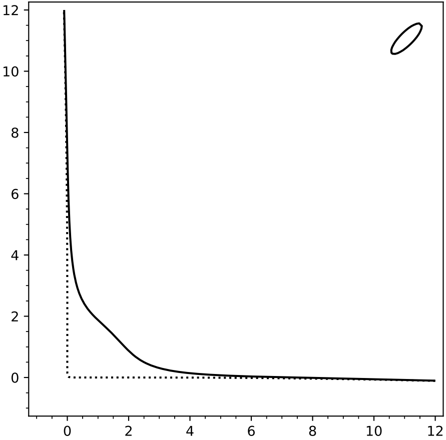


$\epsilon = 0$ $\epsilon = 1$

$N = 5$; $\Delta = \epsilon$



$\epsilon = 1000$



$\epsilon = -1000$

The singular locus of

$$W_N^{(m)}(x_1,\dots x_m,z)=\prod_{j=1}^{N-1}\left(1+\sum_{l=1}^{m-1}Y_j^{(l)}\right)\cdot\left(x_1+\sum_{l=1}^{m-1}\prod_{j=1}^{N-1}\frac{x_{l+1}}{Y_j^{(l)}}\right)-x_0=0$$

is given by

$$x_0x_1x_2x_3\dots x_m\Delta(x_0,x_1,\dots x_m)=0,$$

where Δ is symmetric polynomial given by

$$\Delta(u_0^N,u_1^N\dots u_m^N)=\prod_{\omega_i}^{\omega_i^N=1}\left(u_0+\sum_{i=1}^{m}\omega_i u_i\right), \qquad (1)$$

where ω_i runs independently over the N-th roots of unity.

N-valued groups, Buchstaber-Rees polynomials

N-valued group is a map

$$\mu : X \times X \to \mathrm{Sym}^n(X)$$

$$\mu(x,y) = x \star y = [z_1, z_2, \ldots, z_n], \quad z_k = (x * y)_k$$

Associative, has unity and inverse element

$$\begin{array}{ccccccc}
 & & X \times \mathrm{Sym}^n(X) & \xrightarrow{D\otimes 1} & \mathrm{Sym}^n(X \times X) & & \\
 & \overset{1\otimes\mu}{\nearrow} & & & & \overset{\mu^n}{\searrow} & \\
X \times X \times X & & & & & & \mathrm{Sym}^{n^2}(X) \\
 & \underset{\mu\otimes 1}{\searrow} & & & & \underset{\mu^n}{\nearrow} & \\
 & & \mathrm{Sym}^n(X) \times X & \xrightarrow{1\otimes D} & \mathrm{Sym}^n(X \times X) & &
\end{array}$$

Coset construction: affine mfd with action of the discrete group => multivalued group

Buchstaber-Rees: Consider $\mathbb{C}$ with action of $\mathfrak{S}_N$ (multiplication by N-root of unity)

$$x \star y = \left[\left(x^{1/N} + \chi^r y^{1/N}\right)^n, \quad 1 \le r \le N\right]$$

Corresponding divisor

$$\Delta_N(x, y, z) = (z - z_1)(z - z_2) \ldots (z - z_N), \quad \mu(x, y) = [z_1, z_2, \ldots z_n]$$

coincides with singularities of $W_N(x, y, z)$

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coincides with singularities of $W_N(x, y, z)$

$$N=2 \quad x^2+y^2+z^2-2xy-2yz-2xz$$

$$N=3 \quad (x-z+y)^3+27xyz$$

$$N=4 \quad x^4-4x^3y-4x^3z+6x^2y^2-124x^2yz+6x^2z^2-4y^3x$$
$$-124xy^2z-124xyz^2-4z^3x+y^4-4y^3z+6y^2z^2-4yz^3+z^4$$

Integrality properties: computer experiments

Exotic expansions

$$\left[\frac{d}{dx}\theta_x^{N-1} - \frac{d}{dz}\theta_z^{N-1}\right] K = 0, \quad \left[\frac{d}{dy}\theta_y^{N-1} - \frac{d}{dz}\theta_z^{N-1}\right] K = 0,$$

with initial conditions

$$K_N(0, y, z) = \frac{1}{z - y}$$

Applying Frobenius method we get

$$K(x, y, z) = \sum_{j=0}^{\infty} a_j(y, z) x^j, \quad a_0(y, z) = \frac{1}{z - y}$$

$$a_j(y, z) = \frac{1}{j^N} \frac{d}{dz} \theta_z^{N-1} [a_{j-1}(y, z)] = \frac{1}{j^N} \frac{d}{dz} (\theta_z^{N-1})^j \frac{1}{z - y}$$

N=2

Kernel reads

$$K_2(x,y;z) \simeq \frac{1}{z-y} + \frac{(y+z)}{(z-y)^3}x + \frac{(y^2+4yz+z^2)}{(z-y)^5}x^2 + \\ + \\ \frac{(y^3+9y^2z+9yz^2+z^3)}{(z-y)^7}x^3 + \frac{(y^4+16y^3z+36y^2z^2+16yz^3+z^4)}{(z-y)^9}x^4 + \dots$$

Operator which annihilates series

$$(x^2+y^2+z^2-2xy-2xz-2yz)\frac{d}{dx}K + (x-z-y)K = 0$$

Gives Sonine-Gegenbauer type kernel

N-numerology

Kernel is

$$K^{(N)}(x,y|z) = \sum_{m=0}^{\infty} P_m^{(N)}(y,z)\frac{x^m}{(z-y)^{mN+1}}, \quad \deg P_m^{(N)} = m(N-1).$$

Statement: $P_i(x,y)$ are monic palindromic

$$P_i^{(N)}(y,z) = \sum_{k+n=i(N-1)} {}^{(N)}T_{k,n}^{(i)} y^k z^n, \quad \mathbf{T_{k,n}^{(i)}} = \mathbf{T_{n,k}^{(i)}} \quad \mathbf{T_{i,0}^{(i)}} = \mathbf{T_{0,i}^{(i)}} = \mathbf{1}$$

Coefficients are

$${}^{(N)}T_{k,m(N-1)-k}^{(m)} = \frac{1}{k!}\frac{d^k}{dt^k}\left[(1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j\right] :=$$

$$= k\text{-th coefficient of } (1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j$$

Some of these numbers are known

A181544 for $N=3$

A262014 for $N=4$

$N=2$	$y+z$
	$y^2+4yz+z^2$
	$y^3+9y^2z+9yz^2+z^3$
	$y^4+16y^3z+36y^2z^2+16yz^3+z^4$
	$y^5+25y^4z+100y^3z^2+100y^2z^3+25yz^4+z^5$
	$y^6+36y^5z+225y^4z^2+400y^3z^3+225y^2z^4+36yz^5+z^6$
$N=3$	$y^2+4yz+z^2$
	$y^4+20y^3z+48y^2z^2+20yz^3+z^4$
	$y^6+54y^5z+405y^4z^2+760y^3z^3+405y^2z^4+54yz^5+z^6$
	$y^8+112y^7z+1828y^6z^2+8464y^5z^3+13840y^4z^4+8464y^3z^5+$ $1828y^2z^6+112yz^7+z^8$
	$y^{10}+200y^9z+5925y^8z^2+52800y^7z^3+182700y^6z^4+273504y^5z^5+$ $182700y^4z^6+52800y^3z^7+5925y^2z^8+200yz^9+z^{10}$

Realness and positivity

m	$N = 2$	$N = 3$	$N = 4$
2	(1, 2)	(1, 2)	(1, 8)
3	(1, 6)	(1, 16, 10)	(1, 66, 324, 104)
4	(1, 12, 6)	(1, 48, 198, 56)	(1, 234, 5076, 19304, 11136)
5	(1, 20, 30)	(1, 104, 1176, 2144, 346)	...
6	(1, 30, 90, 20)	(1, 190, 4360, 21200, 21650, 2252)	...

Links and perspectives

Kontsevich (2007-2009): To make explicite Langlands correspondence for SL_2- local systems L on $\mathbb{P}^1_{\mathbb{F}_p} \setminus \mathrm{four \quad points}$ with unipotent local monodromies, which correspond to a special Heun equation

$$L := \partial f \partial + t + \lambda, \quad f = t^3 + at^2 + bt + c$$

- a cubic polynomial (See also Teruji Thomas, 2006 (MSci at U Chicago with V. Drinfeld and Niels uit de Bos, 2019 (PhD U Essen with J. Heinloth.)

Define

$$P(x, y, z) = \mathrm{disc}_t(f(t) - (t - x)(t - y)(t - z))$$

Let $\mathbb{F}_p-$ a finite field with a prime characteristic $p \neq 2$. $x \in \mathbb{P}^1(\mathbb{F}_p)$, $H_x \in \mathrm{Mat}_{(p+1)\times(p+1)}(\mathbb{F}_p)$ such that

$$(Hx)_{yz} := 2 + \sharp\{w | w^2 = P(x, y, z) + \mathrm{correction \quad term}\}$$

A miracle: $[H_x, H_y] = 0$! Operators $H_x, x \in \mathbb{P}^1(\mathbb{F}_p)$ generate *a Hecke algebra* $\mathcal{H}$.

$$\mathcal{H} \curvearrowright \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_p)) : (H_x(\bar{v}))_y = \sum_z (H_x)_{yz}(v_z),\ x, y, z \in P^1(\mathbb{F}_p), (\bar{v}) = (v_x) \in \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_p)).$$

Drinfeld: when $PGL_2(\mathbb{F}_p(t))-$ automorphic representations correspond to normalized common Hecke eigenvectors:

$$H_x(\bar{v}) = v_x \bar{v},\ v_x v_y = \sum_z (H_x)_{yz}(v_z)$$

2-Bessel kernel singularities is a degeneration of the Kontsevich polynomial

Kernel may be considered as an analog of the Hecke operator in the analytic context

Higher Bessel Kernels should correpond to the degeneration of the higher rank local systems

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Non-Abelian Abel's theorem by V. Golyshev, V.R., A. Mellit and D. van Straten:

Kernels are lifts of the Abel's law on a base curve

Example: Kontsevich polynomial - elliptic curve Abelians law

N-Bessel kernels lift Buchstaber-Rees law

Lift from the corresponding spectral curve

$$\delta = x\frac{d}{dx} - A = x\frac{d}{dx} - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ x & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\boxed{\Gamma(x,\lambda) = (-1)^N \det(A/x - \lambda) = \lambda^N - 1/x = 0}$$

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$$\delta = x\frac{d}{dx} - A = x\frac{d}{dx} - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ x & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\boxed{\Gamma(x,\lambda) = (-1)^N \det(A/x - \lambda) = \lambda^N - 1/x = 0}$$

Non-Abelian Abel's theorem

Non-Abelian Abel's theorem by V. Golyshev, V.R., A. Mellit and D. van Straten:

Kernels are lifts of the Abel's law on a base curve

Example: Kontsevich polynomial - elliptic curve Abelians law

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Bessel moments

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$$(\mathcal{P}_k)_{ij} = \int_0^\infty I_0(z)^i K_0(z)^{k-i} z^{2j-1} dz, \quad 1 \le i,j \le \lfloor (k-1)/2 \rfloor$$

Bernoulli matrix

$$(\mathcal{B}_k)_{ij} = (-1)^{k-i} \frac{(k-i)!(k-j)!}{k!} \frac{B_{k-i-j+1}}{(k-i-j+1)!}$$

Quadratic relation

$$\mathcal{P}_k \cdot \mathcal{D}_k \cdot \mathcal{P}_k = (-2\pi i)^k \mathcal{B}_k$$

Conjectured by Broadhurst and Roberts. Proved and extended by Fresán, Sabbah and Yu

Known as Kloosterman motive, framework: Irregular Hodge theory

=> Oscillatory integrals for Dwork superpotentials and corresponding Thom-Sebastiani sums

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$$\int_0^\infty I_0(z)K_0(z)^{l+1}\, z\, dz = \frac{1}{2^l} \int_{X_i \ge 0} \frac{1}{1 - W(X)} \prod_{k=1}^{l} \frac{dX_k}{X_k}, \quad W = \left(1 + \sum_{i=1}^{l+1} X_i\right)\left(1 + \sum_{i=1}^{l+1} X_i^{-1}\right)$$

Same potential as for 2-Bessel multi-product restricted to diagonal, BUT different integration path

Studied as one parametric families

$$(X_0 + X_1 + \ldots + X_m)\left(\frac{x_0}{X_0} + \frac{x_1}{X_1} + \ldots + \frac{x_m}{X_m}\right) = 1/t.$$

Appeared in banana graph Feynman integrals (Bloch, Kerr and Vanhove; Klemm, Dühr). Calabi-Yau case $m = 5$ (Hulek and Verrill; Candelas c.s.)

W_N give a universal differential forms which connects oscillatory integrals in LHS and periods in RHS

Works not only for diagonal

Reminds us Givental's MS which connects

$$\text{Oscillatory Integrals} \quad \leftrightarrow \quad \text{Periods}$$

Should work also for $N > 2$, but need to find a corresponding pairing

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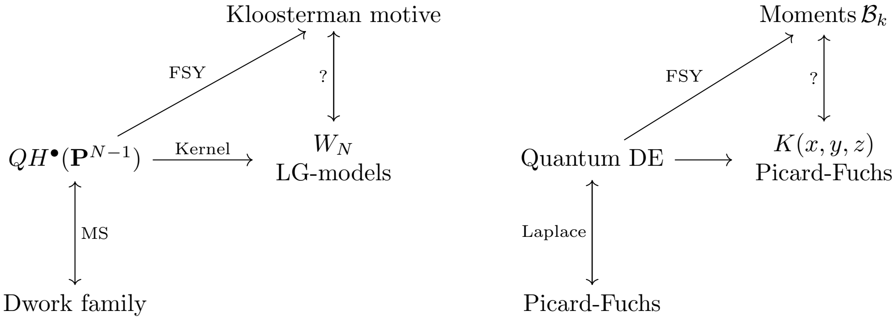

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Kernel
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LG-models
MS
Dwork family
Moments $\mathcal{B}_k$
FSY
?
Quantum DE
$K(x, y, z)$
Picard-Fuchs
Laplace
Picard-Fuchs

Thank you for your time