Connecting W-algebras and their representations

Integrable systems and automorphic forms, Lille

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Motivations

Vertex algebras play an important role in many areas of Mathematics

- moonshine conjectures [Borcherds, Frenkel-Lepowsky-Meurman]
- integrable hierarchies [Drinfeld-Sokolov, De Sole-Kac-Valeri,...]
- geometric Langlands program [Frenkel-Gaitsgory,...]
- instanton moduli spaces, cohomological Hall algebras [Braverman-Finkelberg-Nakajima, Rapčák-Soibelman-Yang-Zhao]

They also appear in Physics in particular in 2-dim CFT and string theory: they formalize the notion of symmetry algebra extending the conformal symmetry (Virasoro algebra).

[Borcherds'86] Vertex algebras can be viewed as a generalization of enveloping algebras for Lie algebras:

vertex algebra V	associative algebra A
Vacuum $\|0\rangle$	Unit 1
$Y(\cdot, z) : V \times V \to V((z))$	Product $\circ : A \times A \to A$
Translation op. ∂	Derivation d

Affine vertex algebras

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be finite dimensional Lie algebra over $\mathbb{C}$ and consider the affine Kac-Moody Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K = \mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}K,$$

$$[K, \widehat{\mathfrak{g}}] = 0 \quad \text{and} \quad [xz^m, yz^n] = [x, y]z^{m+n} + m(x|y)\delta_{m+n,0}K,$$

with $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $(\ |\ ) = \frac{1}{2h^\vee}$Killing and $h^\vee$ the dual Coxeter number.

It is a central extension over the loop algebra:

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g}[z, z^{-1}] \to 0.$$

For $k \in \mathbb{C}$, we associate the affine vertex algebra

$$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[z] \oplus \mathbb{C}K)} \mathbb{C}_k \simeq U(\mathfrak{g} \otimes z^{-1}\mathbb{C}\,[z^{-1}]),$$

where $\mathbb{C}_k$ is a 1-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as $k\,\mathrm{Id}_{\mathbb{C}_k}$.

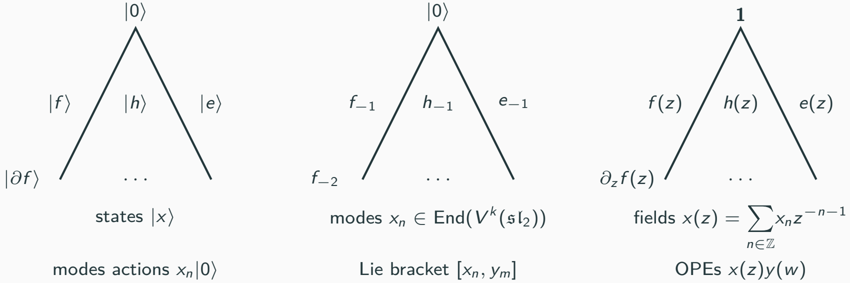
An example: $V^k(\mathfrak{sl}_2)$

Consider $\mathfrak{sl}_2 = \mathrm{Vect}\{e, h, f\}$,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $[x, y] = xy - yx$ $(x, y \in \mathfrak{sl}_2)$.

$V^k(\mathfrak{sl}_2)$: one object, three approaches



Representations & Zhu's algebra

The representation theory of $V^k(\mathfrak{g})$ captures the smooth $\widehat{\mathfrak{g}}$-modules at level k:

$$V^k(\mathfrak{g})\text{-Mod} = \{M \mid a(z)v \in M(\!(z)\!)\} \subset \widehat{\mathfrak{g}}_k\text{-Mod}\,.$$

Moreover, $\mathbb{Z}_{\geqslant 0}$-graded rep., irreducible objects are in one-to-one correspondence with irreducible representations of $\mathrm{Zhu}(V^k(\mathfrak{g})) \simeq U(\mathfrak{g})$:

$$U(\mathfrak{g})\text{-Mod} \longrightarrow V^k(\mathfrak{g})\text{-Mod}, \quad \mathcal{L}(\lambda) \longmapsto \mathcal{L}_k(\lambda).$$

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$$V^k(\mathfrak{g}) \xrightarrow{\text{Zhu}} U(\mathfrak{g}) \xrightarrow{\text{gr}} \mathbb{C}[\mathfrak{g}]$$

Whittaker models

Let $f \in \mathfrak{g}$ nilpotent and consider the finite W-algebra $U(\mathfrak{g}, f)$:

$$U(\mathfrak{g}, f) \simeq (U(\mathfrak{g})/\mathcal{I}_\chi)^{\mathfrak{g}_+} \simeq \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g})/\mathcal{I}_\chi, U(\mathfrak{g})/\mathcal{I}_\chi),$$

where $\mathcal{I}_\chi = \langle a - \chi(a) \mid a \in \mathfrak{g}_+ \rangle$.

$U(\mathfrak{g}, f)$ acts on the (twisted) invariants $M^{G_+,\chi}$ of a $\mathfrak{g}$-module M

$\rightsquigarrow$ Skryabin's equivalence: $\quad U(\mathfrak{g})\text{-Mod}^{G_+,\chi} \xrightarrow{\sim} U(\mathfrak{g}, f)\text{-Mod}.$

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• In the affine setting, definition more subtle [Gaitsgory et al.], but for regular nilpotent f_{reg}, we have an equivalence [Raskin'16]

$$\mathrm{Whit}_{\mathrm{reg}}(V^k(\mathfrak{g})\text{-Mod}) \simeq \text{“}W^k(\mathfrak{g}, f_{\mathrm{reg}})\text{-Mod”}.$$

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• Similarly for local fields ($\mathbb{F} = \mathbb{Q}_p$ or $\mathbb{Z}_p((z))$), they are used to study $G(\mathbb{F})$-modules (take $\mathrm{Fun}(G(\mathbb{F}))$ rather than $U(\mathfrak{g}(\mathbb{F}))$).

In particular, Whittaker models $\mathrm{Whit}_{\mathrm{reg}}(\mathrm{Fun}(\mathrm{SL}_n(\mathbb{F}))) := W_{\mathbb{O}_n} \simeq \mathcal{I}^n(\mathbb{C})$ can be generalized to $\mathbb{O}_\lambda$, $\lambda \in \mathcal{P}(n)$ and [Gomez-Gourevitch-Sahi'17]

$$W_{\mathbb{O}_\lambda} \simeq \mathcal{I}^{\lambda_1} \dots \mathcal{I}^{\lambda_\ell}(\mathbb{C}).$$

Works in that direction for finite W-algebras too [Morgan'14, Genra-Juillard'23].

Conjecture and webs of W-algebras

Conjecture

Si $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}(n)$, $W^k(\mathfrak{sl}_n, f_\lambda) \simeq H_{\lambda_1} H_{\lambda_2} \ldots H_{\lambda_\ell} \left(V^k(\mathfrak{sl}_n)\right)$.

Proved for $N \leqslant 5$ [Creutzig-F.-Linshaw-Nakatsuka'24].

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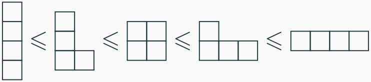
Conjecture supported by Yang-Mills theories:

Vertex algebras can be obtained from higher-dimensional gauge theories along 2D boundaries. In particular, gluing certain 4D Yang-Mills theories with 3D+2D boundaries, we can obtain W-algebras [Gaiotto-Rapčák'18].

[Procházka-Rapčák'18] obtained more W-algebras, starting with webs of interfaces.

W-algebras [Feigin-Frenkel,'90 & Kac-Roan-Wakimoto,'03]

Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{sl}_n$. It is a finite union of disjoint orbits $\mathbb{O}_f := \mathrm{SL}_n.f$ ($f \in \mathcal{N}$) parameterized by the poset $\mathcal{P}(n)$:



$$0 \leqslant \mathbb{O}_{\mathrm{min}} \leqslant \mathbb{O}_{\{2,2\}} \leqslant \mathbb{O}_{\mathrm{sub}} \leqslant \mathbb{O}_{\mathrm{reg}}.$$

To each orbit $\mathbb{O}_\lambda$ corresponds an (affine) W-algebra obtained from $V^k(\mathfrak{sl}_n)$ by the BRST reduction:

$$W^k(\mathfrak{sl}_n, \mathbb{O}_\lambda) = H^0_\lambda(V^k(\mathfrak{sl}_n)).$$

$$\begin{array}{ccccc}
V^k(\mathfrak{sl}_n) & \xrightarrow{\text{Zhu}} & U(\mathfrak{sl}_n) & \xrightarrow{\text{gr}} & \mathbb{C}[\mathfrak{sl}_n] \\
\Big\downarrow H^0_\lambda & & \Big\downarrow (\text{-}/\mathcal{I}_\chi)^{(\mathfrak{sl}_n)_+} & & \Big\downarrow /\!/_\chi (\mathrm{SL}_n)_+ \\
W^k(\mathfrak{sl}_n, \mathbb{O}_\lambda) & \rightarrow & U(\mathfrak{sl}_n, f_\lambda) & \longrightarrow & \mathbb{C}[\mathcal{S}_{f_\lambda}]
\end{array}$$

where $\mathcal{S}_f = f + (\mathfrak{sl}_n)^e$ is the Slodowy slice of f.

Example: $V^k(\mathfrak{sl}_2)$ and Vir^{c_k}

For $V^k(\mathfrak{sl}_2)$, gauge condition by setting $e(z)$ to be a constant. Implemented by constructing a BRST differential ($d = \, :(e(z) + \mathbf{1})\varphi(z):$) and computing its cohomology.

The result is the Virasoro vertex algebra Vir^{c_k}, strongly generated by the field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfying the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\partial_{m+n,0}c_k,$$

where c_k is the central charge.

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$$\begin{array}{ccccc}
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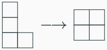
\mathrm{Vir}^{c_k} & \longrightarrow & Z(\mathfrak{sl}_2) \simeq \mathbb{C}[c] & \longrightarrow & \mathbb{C}[x]
\end{array}$$

Reduction of the affine subalgebra [Creutzig-F.-Linshaw-Nakatsuka'24]

Example with $\mathfrak{sl}_4$: 

For almost all $k \in \mathbb{C}$, we have conformal embeddings

$$\begin{aligned} \langle L, W_3 \rangle \otimes V^{k+1}(\mathfrak{sl}_2) \otimes \mathcal{H} &\hookrightarrow W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}}) \\ \langle L, W_3 \rangle \otimes \mathrm{Vir}^{c_{k+1}} \otimes \mathcal{H} &\hookrightarrow W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,2\}}), \end{aligned}$$

where $\langle L, W_3 \rangle \simeq \langle L, W_3 \rangle \simeq \mathrm{Com}\left(V^{k+1}(\mathfrak{gl}_2), W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}})\right)$ when $k \notin \mathbb{Q}$.

Apply H_2^0 to the affine part of $W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}})$ gives

$$H_2^0(W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}})) \simeq W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,2\}}).$$

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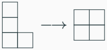
Example with $\mathfrak{sl}_4$: 

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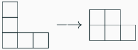
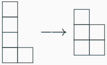
Works similarly with $\mathfrak{sl}_5$:  and 

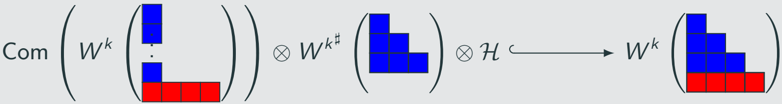
$$H_2^0(W^k(\mathfrak{sl}_5, \mathbb{O}_{\{3,1^2\}})) \simeq W^k(\mathfrak{sl}_5, \mathbb{O}_{\{2,3\}})$$

$$H_{\{2,1\}}^0(W^k(\mathfrak{sl}_5, \mathbb{O}_{\{2,1^3\}})) \simeq W^k(\mathfrak{sl}_5, \mathbb{O}_{\{2^2,1\}}).$$

Structure of W-algebras

Conjecture [CFLN'24]

Let $\lambda = \{\lambda_1, \dots, \lambda_{\ell-1}, \lambda_\ell\} \in \mathcal{P}(n)$, then there is a conformal embedding



Iterating, we get that W-algebras decompose as product of affine cosets

$$\mathrm{Com}\left(V^{k^\sharp}(\mathfrak{gl}_m), W^k(\mathfrak{sl}_n, f_{\{n-m,1^m\}})\right).$$

Surprisingly, affine cosets are all obtain as quotients of the same W_∞-algebra $W(2,3,\dots,)$ [Linshaw'21], which is conjecturally isomorphic to the affine Yangian of $\mathfrak{gl}_1 \simeq \mathbb{C}$.

Hamiltonian reductions and representations

Recall: $\mathbb{Z}_{\geqslant 0}$-graded $V^k(\mathfrak{sl}_n)$-modules are the same as modules of $U(\mathfrak{sl}_n)$.

Rep. theory of the simple quotient $V_k(\mathfrak{sl}_n)$ of $V^k(\mathfrak{sl}_n)$ is more complicated.

When $k \in \mathbb{Z}_{\geqslant 0}$, $V_k(\mathfrak{sl}_n)$ is rational: category of hw modules is finite and semisimple, simple objects are the integrable rep. [Frenkel-Zhu'92].
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Ex: for $V_k(\mathfrak{sl}_2)$, $\{\mathcal{L}_k(\lambda), \lambda \in [\![0, k]\!]\}$

Fusion rules:

$V_0(\mathfrak{sl}_2)$	0
0	0

$V_1(\mathfrak{sl}_2)$	0	ϖ_1
0	0	ϖ_1
ϖ_1	ϖ_1	0

$V_2(\mathfrak{sl}_2)$	0	ϖ_1	$2\varpi_1$
0	0	ϖ_1	$2\varpi_1$
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$2\varpi_1$	$2\varpi_2$	ϖ_1	0

When k is admissible (i.e. $k + n = p/q$, $p \geqslant n$, $q \geqslant 1$) not in $\mathbb{Z}$:
Simple hw $V_k(\mathfrak{sl}_n)$-modules are admissible rep. They still satisfy modularity properties [Kac-Wakimoto'88] but they are not stable under fusion product.

$\rightsquigarrow$ modularity explained by rationality of the simple W-algebra $W_k(\mathfrak{sl}_n, \mathbb{O}_{\{q^s,r\}})$.

Relaxed modules

For $k = -n + p/q$ admissible, $W_k(\mathfrak{sl}_n, \mathbb{O}_{\{q^s,r\}})$ is rational [Arakawa-van Ekeren'19] and $H^0_\lambda : V_k(\mathfrak{sl}_n)$-Mod $\to W_k(\mathfrak{sl}_n, \mathbb{O}_\lambda)$-Mod is a surjective functor that maps irreducible on irreducible (or 0).

However, it is not injective so we need to start with a bigger category of modules in $V_k(\mathfrak{sl}_n)$.

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Ex: for $V_k(\mathfrak{sl}_2)$, $k = -2 + \frac{p}{q}$, $(p, q \geqslant 2)$

[Creutzig-Ridout'13, Kawasetsu-Ridout'19]: relaxed $V_k(\mathfrak{sl}_2)$-modules generate a modular tensor category.

Inverse Hamiltonian reduction

$$\mathrm{ch}[\mathcal{E}_k(\lambda)](z;q) = \sum z^{h_0} q^{L_0} \dim[\mathcal{E}_k(\lambda)]_{h_0,L_0} = z^{\lambda} \frac{\mathrm{ch}[\mathcal{L}_{c_k}(\Delta_{\lambda,k})](q)}{\eta(q)^2} \delta(z^{\alpha}),$$

where $\mathcal{L}_{c_k}(\Delta_{\lambda,k})$ is a Virasoro minimal model [Kawasetsu-Ridout'19].

For $c_k = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$, Vir_{c_k} is rational [Wang'93]. Its irred. $\mathbb{Z}_{\geqslant 0}$-graded rep. are exactly the Virasoro minimal models $\{\mathcal{L}_{c_k}(\Delta_{\lambda,k})\}$.

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There is a conformal embedding [Semikhatov'94]

$$V^k(\mathfrak{sl}_2) \hookrightarrow \mathrm{Vir}^{c_k} \otimes \Pi_{\mathbb{Z}}$$

$$e(z) \mapsto e^{c}(z), \quad h(z) \mapsto \frac{k}{2}c(z) + d(z), \quad f(z) \mapsto (k+2) : L(z)e^{-c}(z) : + \dots$$

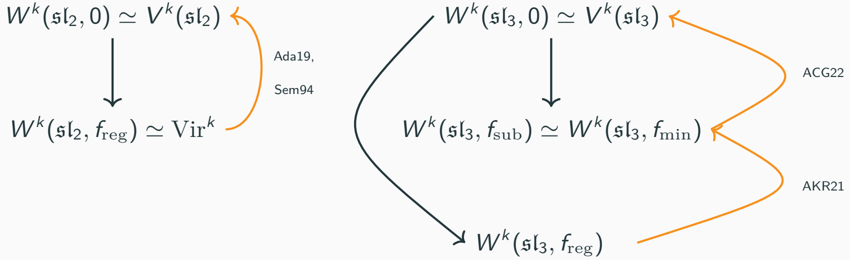
preserved by taking simple quotients for $k = -2 + \frac{p}{q} \notin \mathbb{Z}$ admissible [Adamović'17].

Relaxed $V_k(\mathfrak{sl}_2)$-modules are reconstructed from $\mathcal{L}_{c_k}(\Delta_{\lambda,k})$:

$$\mathcal{E}_k(\lambda) \simeq \mathcal{L}_{c_k}(\Delta_{\lambda,k}) \otimes \Pi[\lambda], \qquad [\lambda] \in \mathbb{C}/\mathbb{Z}.$$

In particular, modularity of relaxed $V_k(\mathfrak{sl}_2)$-modules is deduced from modularity of Virasoro minimal models.

More modularity & inverse reductions



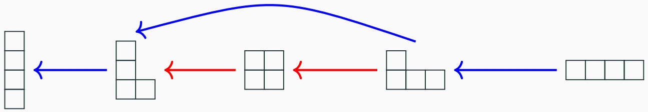
Using explicit embeddings, modularity of relaxed modules has been checked for $k = -3 + p/q$ admissible:

- [Fehily-Ridout'22]: $W_k(\mathfrak{sl}_3, f_{\mathrm{min}})$, $q \geqslant 3$ i.e. $W_k(\mathfrak{sl}_3, f_{\mathrm{reg}})$ rational,
- [F.-Raymond-Ridout'24]: $V^k(\mathfrak{sl}_3)$, $q = 2$ i.e. $W_k(\mathfrak{sl}_3, f_{\mathrm{min}})$ rational, using

$$V^k(\mathfrak{sl}_3) \hookrightarrow W^k(\mathfrak{sl}_3, \mathbb{O}_{\mathrm{sub}}) \otimes \beta\gamma \otimes \Pi_{\mathbb{Z}}$$ [Adamović-Creutzig-Genra'22].

Generalizing inverse Hamiltonian reductions

Nilpotent orbits of $\mathfrak{sl}_n$ are partially ordered:

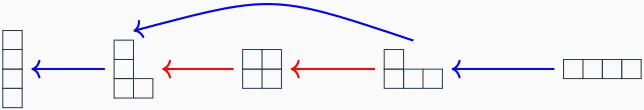


Idea: If $\mathbb{O}_\lambda \leqslant \mathbb{O}_{\lambda'}$ (+ conditions) then

$$W^k(\mathfrak{sl}_n, \mathbb{O}_\lambda) \hookrightarrow W^k(\mathfrak{sl}_n, \mathbb{O}_{\lambda'}) \otimes \text{free fields}.$$

Generalizing inverse Hamiltonian reductions

Nilpotent orbits of $\mathfrak{sl}_n$ are partially ordered:



Idea: If $\mathbb{O}_\lambda \leqslant \mathbb{O}_{\lambda'}$ (+ conditions) then

$$W^k(\mathfrak{sl}_n, \mathbb{O}_\lambda) \longrightarrow W^k(\mathfrak{sl}_n, \mathbb{O}_{\lambda'}) \otimes \text{free fields}.$$

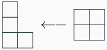
Theorem [Fehily'23]

IHR between hook-type partitions. For $m' \leqslant m \leqslant n$,

$$W^k(\mathfrak{sl}_n, \mathbb{O}_{\{m', 1^{n-m'}\}}) \hookrightarrow W^k(\mathfrak{sl}_n, \mathbb{O}_{\{m, 1^{n-m}\}}) \otimes V$$

where $V = \beta\gamma^{\otimes a} \otimes \Pi_{\mathbb{Z}}^{\otimes b}$.

Inverting the partial reduction

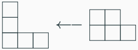
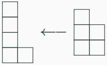
Example with $\mathfrak{sl}_4$: 

For almost all $k \in \mathbb{C}$, we have conformal embeddings

$$\langle L, W_3 \rangle \otimes V^{k+1}(\mathfrak{sl}_2) \otimes \mathcal{H} \hookrightarrow W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}})$$
$$\langle L, W_3 \rangle \otimes \mathrm{Vir}^{c_{k+1}} \otimes \mathcal{H} \hookrightarrow W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,2\}}),$$

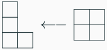
Extend $V^{k+1}(\mathfrak{sl}_2) \hookrightarrow \mathrm{Vir}^{c_{k+1}} \otimes \Pi_{\mathbb{Z}}$ gives

$$W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,1^2\}}) \hookrightarrow W^k(\mathfrak{sl}_4, \mathbb{O}_{\{2,2\}}) \otimes \Pi_{\mathbb{Z}}.$$

Similarly with $\mathfrak{sl}_5$:  and 

$$W^k(\mathfrak{sl}_5, \mathbb{O}_{\{3,1^2\}}) \hookrightarrow W^k(\mathfrak{sl}_5, \mathbb{O}_{\{3,2\}}) \otimes \Pi_{\mathbb{Z}}.$$
$$W^k(\mathfrak{sl}_5, \mathbb{O}_{\{2,1^3\}}) \hookrightarrow W^k(\mathfrak{sl}_5, \mathbb{O}_{\{2^2,1\}}) \otimes \beta\gamma \otimes \Pi_{\mathbb{Z}}.$$

Inverting the partial reduction

Example with $\mathfrak{sl}_4$: 

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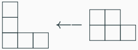
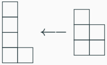
Similarly with $\mathfrak{sl}_5$:  and 

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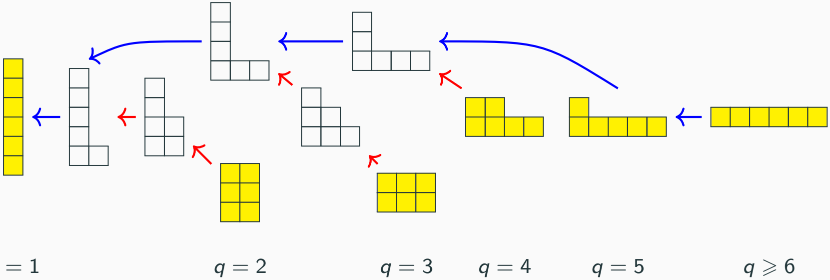
Conjecture

For $\lambda = \{\lambda_1 \geqslant \ldots \geqslant \lambda_\ell\} \in \mathcal{P}(n)$. There is a conformal embedding

$$W^k(\mathfrak{sl}_n, \mathbb{O}_{\{\lambda_1, \cdots, \lambda_p, 1^m\}}) \hookrightarrow W^k(\mathfrak{sl}_n, \mathbb{O}_{\{\lambda_1, \cdots, \lambda_p+1, 1^{m-1}\}}) \otimes \Pi_{\mathbb{Z}} \otimes \beta\gamma^{\otimes(m-2)}.$$

General pattern & Rationality



Partial/inverse reductions provide paths to connect $V_k(\mathfrak{sl}_n)$ to rational W-algebras.

Actually, we have more connections of W-algebras coming from nilpotent orbit closures relation [Beem-Buston-Nair, Genra-Juillard, F.-Fehily-Fursman-Nakatsuka, works in progress].

More partial/inverse reductions [F.-Fehily-Fursman-Nakatsuka]

$W^k(\mathfrak{sl}_3, 0) \simeq V^k(\mathfrak{sl}_3)$

$\downarrow$

$W^k(\mathfrak{sl}_3, f_{\mathrm{sub}}) \simeq W^k(\mathfrak{sl}_3, f_{\mathrm{min}})$

$\downarrow$ Madsen Ragoucy '97

$W^k(\mathfrak{sl}_3, f_{\mathrm{reg}})$

$W^k(\mathfrak{sl}_4, 0) \simeq V^k(\mathfrak{sl}_4)$

$\downarrow$

$W^k(\mathfrak{sl}_4, f_{\mathrm{min}})$

$\downarrow$ CFLN'24

$W^k(\mathfrak{sl}_4, f_{2,2})$

$\downarrow$ FFFN

$W^k(\mathfrak{sl}_4, f_{\mathrm{sub}})$

$\downarrow$ FFFN

$W^k(\mathfrak{sl}_4, f_{\mathrm{reg}})$

$\rightarrow$ (orange): reduction of affine subalgebra.

$\rightarrow$ (blue): reduction of a natural representation for the affine subalgebra.

Ex: $W^k(\mathfrak{sl}_3, f_{\mathrm{sub}})$ is strongly generated by fields H, L, G^+, G^-.

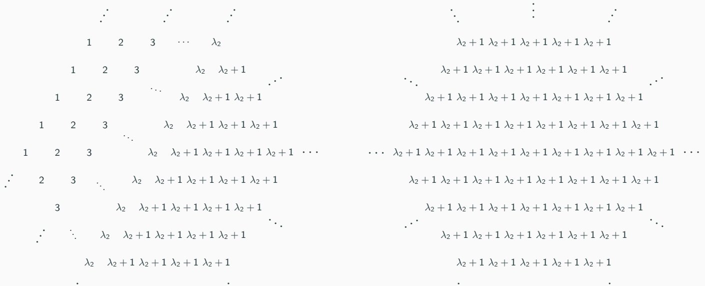
G^+ gen. a 1-dim natural rep. of $H \rightsquigarrow H^0_2(W^k(\mathfrak{sl}_3, f_{\mathrm{sub}})) \simeq W^k(\mathfrak{sl}_3, f_{\mathrm{reg}})$.

Relaxed modules for $V_k(\mathfrak{sl}_3)$

$$V_k(\mathfrak{sl}_3) \hookrightarrow W_k(\mathfrak{sl}_3, \mathbb{O}_{\mathrm{sub}}) \otimes \beta\gamma \otimes \Pi_{\mathbb{Z}}$$

For $q = 2$, i.e. $k = -3 + \frac{p}{2}$ with $p = 3, 5, \ldots$, $W_k(\mathfrak{sl}_3, \mathbb{O}_{\mathrm{sub}})$ is rational.

	$W_k(\mathfrak{sl}_3, f_{\mathrm{sub}})$	$\beta\gamma$	Π
$\widehat{\mathcal{S}}_{\lambda,[\nu]}$	hw mod $H^0_{f_{\mathrm{sub}}}(L(\widehat{\lambda}))$ $(\lambda \in \mathrm{P}^{u-3}_{+})$	hw mod $\mathcal{V}$	relaxed mod $\Pi_{[\nu]}$ $([\nu] \in \mathbb{C}/\mathbb{Z})$
$\widehat{\mathcal{R}}_{\lambda,[\mu,\nu]}$	hw mod $H^0_{f_{\mathrm{sub}}}(L(\widehat{\lambda}))$ $(\lambda \in \mathrm{P}^{u-3}_{+})$	relaxed mod $\mathcal{W}_{[\mu]}$ $([\mu] \in \mathbb{C}/\mathbb{Z}\backslash\{[0]\})$	relaxed mod $\Pi_{[\nu]}$ $([\nu] \in \mathbb{C}/\mathbb{Z})$



$k = -3 + \frac{p}{q}$ with $q \geqslant 3$

$W_k(\mathfrak{sl}_3, \mathbb{O}_{\text{sub}})$	$\beta\gamma$	$\Pi_{\mathbb{Z}}$
hw mod $H^0_{f_{\text{sub}}}(L(\widehat{\lambda}))$ $(\lambda \in \mathrm{P}^{u-3}_{+})$	hw mod $\mathcal{V}$	rel mod $\Pi_{[\nu]}$ $([\nu] \in \mathbb{C}/\mathbb{Z})$
hw mod $H^0_{f_{\text{sub}}}(L(\widehat{\lambda}))$ $(\lambda \in \mathrm{P}^{u-3}_{+})$	rel mod $\mathcal{W}_{[\mu]}$ $([\mu] \in \mathbb{C}/\mathbb{Z} \backslash \{[0]\})$	rel mod $\Pi_{[\nu]}$ $([\nu] \in \mathbb{C}/\mathbb{Z})$
rel mod	...	...

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rel⊗rel⊗rel